



Symbolic Automata: Omega-Regularity Modulo Theories

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Symbolic automata are finite state automata that support potentially infinite alphabets, such as the set of rational numbers, generally applied to regular expressions and languages over finite words. In symbolic automata (or automata modulo \mathcal{A}), an alphabet is represented by an effective Boolean algebra \mathcal{A} , supported by a decision procedure for satisfiability. Regular languages over infinite words (so called ω -regular languages) have a rich history paralleling that of regular languages over finite words, with well known applications to model checking via Büchi automata and temporal logics.

We generalize symbolic automata to support ω -regular languages via *transition terms* and *symbolic derivatives*, bringing together a variety of classic automata and logics in a unified framework that provides all the necessary ingredients to support symbolic model checking modulo \mathcal{A} . In particular, we define: (1) alternating Büchi automata modulo \mathcal{A} ($ABW_{\mathcal{A}}$) as well (non-alternating) nondeterministic Büchi automata modulo \mathcal{A} ($NBW_{\mathcal{A}}$); (2) an alternation elimination algorithm \mathcal{A} that incrementally constructs an $NBW_{\mathcal{A}}$ from an $ABW_{\mathcal{A}}$, and can also be used for constructing the product of two $NBW_{\mathcal{A}}$; (3) a definition of linear temporal logic modulo \mathcal{A} , $LTL\langle\mathcal{A}\rangle$, that generalizes Vardi's construction of alternating Büchi automata from LTL, using (2) to go from LTL modulo \mathcal{A} to $NBW_{\mathcal{A}}$ via $ABW_{\mathcal{A}}$.

Finally, we present $RLTL\langle\mathcal{A}\rangle$, a combination of $LTL\langle\mathcal{A}\rangle$ with extended regular expressions modulo \mathcal{A} that generalizes the Property Specification Language (PSL). Our combination allows *regex complement*, that is not supported in PSL but can be supported naturally by using transition terms. We formalize the semantics of $RLTL\langle\mathcal{A}\rangle$ using the *Lean* proof assistant and formally establish correctness of the main derivation theorem.

CCS Concepts: • **Theory of computation** → **Automata over infinite objects; Regular languages; Modal and temporal logics; Verification by model checking;** • **Computing methodologies** → **Symbolic and algebraic manipulation.**

Additional Key Words and Phrases: regular expression, temporal logic, derivative, satisfiability modulo theories, automata

ACM Reference Format:

Margus Veanes, Thomas Ball, Gabriel Ebner, and Ekaterina Zhuchko. 2025. Symbolic Automata: Omega-Regularity Modulo Theories. *Proc. ACM Program. Lang.* 9, POPL, Article 2 (January 2025), 34 pages. <https://doi.org/10.1145/3704838>

1 Introduction

Classical finite automata, which correspond to regular expressions and regular languages, are finite in *three orthogonal dimensions*:

- ✓ the number of *states* in the automata, which make it a *finite* (or *finite-state*) automata;

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ACM 2475-1421/2025/1-ART2

<https://doi.org/10.1145/3704838>

- ✓ the number of *alphabet elements* that induce state transitions;
- ✓ the length of the *words* (a sequence of alphabet elements) recognized by the automata.

Symbolic automata [D’Antoni and Veanes 2021] are finite state automata that support potentially infinite alphabets, such as the set of rational numbers. The transitions of symbolic automata are labeled with predicates from an effective Boolean algebra (EBA) \mathcal{A} . Each atomic predicate and their Boolean combination in \mathcal{A} can compactly represent a (possibly infinite) subset of elements.

The power of symbolic automata is to make use of decision procedures for \mathcal{A} [de Moura and Bjørner 2011] to avoid the need for reduction to the propositional (classic) case, which can incur an exponential blowup. For example, a propositional encoding of a Boolean expression of size N in \mathcal{A} would require $O(2^N)$ conjunctions of atomic predicates in a propositional setting (so called “mintermization”). In some cases, a classical algorithm for finite automata can be lifted straightforwardly to symbolic automata; in other cases, entirely new algorithms are required [D’Antoni and Veanes 2021].

Regular languages over infinite words (so called ω -regular languages) have a rich history paralleling that of regular languages over finite words; In fact, the two are intimately connected: [McNaughton 1966] showed that ω -regular languages are equivalently described by Büchi automata and the union of a finite set of ω -regular expressions, where each expression has the form $\alpha_i \beta_i^\omega$, where α_i and β_i are regular expressions for finite words and β_i is infinitely repeated using the ω -closure operator. Alternating Büchi automata have equivalent expressive power as (nondeterministic) Büchi automata, but can be exponentially more compact [Miyano and Hayashi 1984].

Furthermore, ω -regular languages are an important part of the theoretical foundation for model checking of reactive systems [Clarke et al. 1999; Pnueli 1985], where both the system and (many of) the properties to be verified of the system are described by sets of infinite traces. Various forms of automata, including Büchi automata, can be used to describe the system while temporal logics describe the properties to be checked. In particular, Linear temporal logic (LTL) [Pnueli 1977] is used to describe the expected behavior of reactive systems and can be lowered to Büchi automata via a variety of algorithms. As LTL does not fully capture ω -regular languages [Emerson 1991; Wolper 1983], various extensions of LTL have been proposed [Armoni et al. 2002; Banieqbal and Barringer 1989; Duret-Lutz 2024; Eisner and Fisman 2006; Sistla et al. 1987; Vardi and Wolper 1994] to increase its expressivity. The LTL-based automata-theoretic approach to model checking constructs the product of the system automaton with the Büchi automaton representing the negation of the LTL formula and then analyzes whether the resulting automaton has any accepting runs [Vardi and Wolper 1986]. If there are no accepting runs then the system automaton is verified with respect to the LTL formula.

Extensions of LTL based program analysis, synthesis, and realizability, that incorporate *modulo theories* reasoning is an active research area. *Fairness modulo theory* [Dietsch et al. 2015] introduces *Büchi programs* whose fair and feasible traces represent feasible traces of a given program that violate a classical LTL property, where feasibility uses SMT solving for satisfiability checking of constraints involving linear arithmetic. *Temporal Stream Logic* [Finkbeiner et al. 2019] (TSL) is an extension of LTL intended for *synthesis* by introducing updates and expressions that may involve arbitrary *uninterpreted* function and predicate symbols. TSL provides increased scalability at the cost of decidability – synthesis for TSL is undecidable in general, while both realizability and synthesis are 2ExpTime-complete for LTL [Pnueli and Rosner 1989]. Further extension of TSL with modulo theories [Finkbeiner et al. 2022] investigates use of *interpreted* functions and predicate symbols, where satisfiability of TSL modulo several standard decidable theories is shown to be neither semidecidable nor co-semidecidable. Recent work on *LTL modulo theories* for realizability [Rodríguez

and Sánchez 2023] uses Boolean propositions in place of theory literals in combination with additional theory constraints, that are checked by an SMT solver in order to guarantee consistent interpretations with the theory literals they stand for. This is a form of *mintermization* through *bitblasting* (see Section 1.3) that we believe can be avoided by a closer integration of reasoning modulo theories into the foundations of linear temporal logic through the theory and algorithms of *symbolic derivatives* that utilize *transition terms*.

1.1 Transition Terms

We generalize symbolic automata to ω -regular languages by first lifting the concept of *transition regexes* from [Stanford et al. 2021] to *transition terms*, which are parametric over two domains:

- ✓ *alphabet*, with an EBA \mathcal{A} over universe Σ , over it;
- ✓ *leaves*, which represent the language of the automata, with its own EBA \mathcal{B} .

Transition terms for \mathcal{B} (modulo \mathcal{A}) are nested if-then-else (ITE) expressions $(\alpha ? f : g)$ whose conditions α come from \mathcal{A} and leaves come from \mathcal{B} , with the intuition “if α then f else g ”. In the case of nested ITE expressions such as $(\alpha ? (\beta ? f : g) : h)$, the primary role of \mathcal{A} is to maintain their *cleanness* by removing unreachable subterms, e.g., if α implies β then g is unreachable.

The EBA \mathcal{B} is typically accompanied with *weak equivalence* laws that are used as rewrite rules to simplify leaves. In particular, conjunction and disjunction are treated as associative, commutative and idempotent (ACI) operations, the law of excluded middle is applied, and the bottom and top elements of \mathcal{B} act as units/zeros of the corresponding Boolean operations. The algebra of transition terms thus becomes an EBA with its own set of rewrite rules derived from \mathcal{A} and \mathcal{B} .

Transition terms can be used as the basis of defining the semantics of languages/logics and their corresponding automata using *symbolic derivatives*, providing a unifying framework for automated reasoning and rewriting that operates incrementally over a symbolic representation.

1.2 Symbolic Derivatives

Let \mathcal{B} be an EBA of formulas ϕ modulo an alphabet EBA \mathcal{A} such that \mathcal{B} is associated with a formal semantics $w \models \phi$ where w is a word over Σ , such that \models respects the Boolean operators of \mathcal{B} . (I.e., $w \models \neg\phi \Leftrightarrow w \not\models \phi$, $w \models \phi \wedge \psi \Leftrightarrow w \models \phi \wedge w \models \psi$, and $w \models \phi \vee \psi \Leftrightarrow w \models \phi \vee w \models \psi$.) We say that \mathcal{B} has a *derivative function* ρ when

$$\text{for all } a \in \Sigma, \rho(\phi, a) \text{ is a formula in } \mathcal{B} \text{ such that } aw \models \phi \Leftrightarrow w \models \rho(\phi, a).$$

Two classical examples of such \mathcal{B} over a *finite* alphabet Σ are *extended regular expressions* (ERE) with $\rho(\phi, a)$ as the derivative $D_a\phi$ in [Brzozowski 1964], and *linear temporal logic* (LTL) with ρ as the transition function ρ in [Vardi 1995]. In the former case $w \in \Sigma^*$ and in the latter case $w \in \Sigma^\omega$.

The *symbolic derivative function* of \mathcal{B} is a function ϱ that maps ϕ into a transition term $\varrho(\phi)$ for \mathcal{B} modulo \mathcal{A} . The transition term $\varrho(\phi)$, called the *symbolic derivative* of ϕ , is such that,

$$\text{for all } a \in \Sigma, \varrho(\phi)[a] = \rho(\phi, a).$$

where $\varrho(\phi)[a]$ returns the leaf that the transition term $\varrho(\phi)$ evaluates to for a . Intuitively, $\varrho(\phi)$ represents a *partial evaluation* of ρ with respect to ϕ . Observe that the size of $\varrho(\phi)$ is independent of the size of Σ (Σ can be *infinite*). Existence of $\varrho(\phi)$ depends on ϕ distinguishing only finitely many conditions of \mathcal{A} , which holds both for ERE modulo \mathcal{A} as well as LTL modulo \mathcal{A} .

An example of a symbolic derivative of a regex $\backslash d^+$ is $(\backslash d ? \backslash d^* : \perp)$ whose condition $\backslash d$ is the character class of all digits. In this case Σ is the set of all characters, that may be a very large set such as Unicode, and the predicates in \mathcal{A} are character classes. E.g., $(\backslash d ? \backslash d^* : \perp)[\text{'7'}] = \backslash d^*$.

An example of an LTL (modulo SMT theory \mathcal{A}) or $\text{LTL}\langle\mathcal{A}\rangle$ formula ϕ and $\varrho(\phi)$ are as follows

$$\begin{aligned}\phi &= \mathbf{G}(0 < x) \wedge ((2|x) \mathbf{U} (3|x)) \\ \varrho(\phi) &= ((0 < x) ? ((3|x) ? \mathbf{G}(0 < x) : ((2|x) ? \phi : \perp)) : \perp)\end{aligned}$$

where x has integer type and ϕ states that x is always positive and remains even until divisible by 3. For example, $\varrho(\phi)[x \mapsto 8] = \phi$ because 8 is a positive even integer that is not divisible by 3. Observe that in this case Σ contains the *infinite* set of all possible interpretations for x .

Integration of symbolic derivatives for ERE into the *sequence theory solver* of Z3 contributed to a significant performance improvement of Z3 [Stanford et al. 2021]. A variant of symbolic derivatives for *regex matching* is also integrated into the new *nonbacktracking* regex matcher of .NET [Moseley et al. 2023] where algebraic rewrites of regexes play a key role as simplification rules during incremental exploration of the state space of a DFA that arises during match search.

1.3 Why Modulo Theories?

The benefits of deeply integrating SMT into the core algorithms are substantial [de Moura and Bjørner 2011] by providing *incremental theory specific reasoning*. At the same time, it has been widely recognized that full ω -regularity is necessary for general applications of linear temporal properties [Pnueli 1985]. Here we achieve both goals modulo \mathcal{A} . Our motivating application is *trace analysis of cloud services* (see Section 7.4). We highlight both aspects next.

1.3.1 Avoiding Mintermization. Consider any $\text{LTL}\langle\mathcal{A}\rangle$ formula ψ that contains the \mathcal{A} predicates $\{\alpha_i\}_{i=1}^n$. A standard technique to represent ψ in classical LTL, is to replace each α_i by a proposition p_i and to add the following theory constraint as a conjunction: $\mathbf{G}(\bigvee\{p_I \mid I \subseteq \{1, \dots, n\}, \text{SAT}(\alpha_I)\})$ where $\alpha_I = (\bigwedge_{i \in I} \alpha_i) \wedge (\bigwedge_{i \notin I} \neg \alpha_i)$ and $p_I = (\bigwedge_{i \in I} p_i) \wedge (\bigwedge_{i \notin I} \neg p_i)$. This is a form of *bitblasting*, with each p_i acting as an independent bit, and where $\text{SAT}(\beta)$ decides satisfiability of the formula β in \mathcal{A} . Irrespective of how this is done, in general, the *cost is bound to be exponential* in n .

1.3.2 ω -Regularity Modulo \mathcal{A} . We introduce a powerful extension $\text{RLTL}\langle\mathcal{A}\rangle$ (RLTL) of $\text{LTL}\langle\mathcal{A}\rangle$ with regexes in ERE modulo \mathcal{A} . Besides increasing the expressive power to be ω -regular modulo \mathcal{A} , it also enables a variety of transformations to utilize regexes. In particular, RLTL includes formulas of the form $R \Box \rightarrow \phi$ called *suffix implications* where R is a regex and ϕ any formula in RLTL , with the semantics $w \models R \Box \rightarrow \phi$ iff $\forall i : w_{..i} \in \mathcal{L}(R) \Rightarrow w_{i..} \models \phi$, where $w_{..i}$ is the finite prefix of w up to position i and $w_{i..}$ is the infinite suffix of w starting from position i . RLTL also includes the dual construct $R \Diamond \rightarrow \phi$ called *existential suffix implication* that is equivalent to $\neg(R \Box \rightarrow \neg\phi)$.

New Superpower. RLTL supports *complement* $\sim R$ of regexes R , such that $v \in \mathcal{L}(\sim R) \Leftrightarrow v \notin \mathcal{L}(R)$. Thus, expressing an ω -regular property such as *all proper prefixes of an infinite word must be in $\mathcal{L}(R)$* becomes elegantly expressible by the formula $\sim R \Box \rightarrow \perp$. Since \perp (false) is unsatisfiable, it follows that $w \models \sim R \Box \rightarrow \perp$ iff $\nexists i : w_{..i} \notin \mathcal{L}(R)$ iff $\forall i : w_{..i} \in \mathcal{L}(R)$.

Combined with intersection and union, this lifts the role of regexes in $\text{RLTL}\langle\mathcal{A}\rangle$ into a self-contained sub-logic whose derivative rules are seamlessly integrated with the derivative rules of the temporal formulas (see Section 7.3). We can therefore leverage algorithms in SMT [Stanford et al. 2021] by lifting them to reason over infinite sequences in temporal logic, while ensuring decidability modulo any core element theory \mathcal{A} that is supported by SMT (e.g., linear arithmetic).

Regex Based Rewrites. Rewriting LTL formulas by introducing regexes can be beneficial by leveraging regex derivatives that can avoid subsequent alternation elimination from an initial Büchi automaton. E.g., the formula $\mathbf{F}\alpha \wedge \mathbf{F}\beta$ where both α and β belong to \mathcal{A} states that eventually α holds and eventually β holds. In RLTL it is equivalent to $\phi = (\tau^* \cdot \alpha \cdot \tau^*) \cap (\tau^* \cdot \beta \cdot \tau^*) \Diamond \rightarrow \top$ where \cap is regex intersection. Then $w \models \phi$ iff some finite prefix of w is in $\mathcal{L}(\tau^* \cdot \alpha \cdot \tau^*) \cap \mathcal{L}(\tau^* \cdot \beta \cdot \tau^*)$.

1.4 Contributions and Overview

We show how transition terms and derivatives can be used to generalize symbolic automata to work with ω -regular languages modulo an infinite alphabet represented by \mathcal{A} , bringing together a variety of classic automata and logics in a unified framework. In particular, we give:

- ✓ a definition of alternating Büchi automata modulo \mathcal{A} ($ABW_{\mathcal{A}}$) and their relationship to classical alternating Büchi automata as well as to (non-alternating) nondeterministic Büchi automata modulo \mathcal{A} ($NBW_{\mathcal{A}}$) (Section 4);
- ✓ an alternation elimination algorithm, a symbolic generalization of [Miyano and Hayashi 1984] that incrementally constructs an $NBW_{\mathcal{A}}$ from an $ABW_{\mathcal{A}}$ (Section 5);
- ✓ a definition of linear temporal logic (LTL) [Pnueli 1977] modulo \mathcal{A} that generalizes Vardi's construction of alternating Büchi automata from LTL [Vardi 1995] – with the previous results, this gives a way to go from LTL modulo \mathcal{A} to $NBW_{\mathcal{A}}$ (Section 6);
- ✓ a combination of LTL modulo \mathcal{A} with extended regular expressions modulo \mathcal{A} ($RLTL\langle\mathcal{A}\rangle$) that generalizes SPOT [Duret-Lutz et al. 2022] and PSL [Eisner and Fisman 2006] (Section 7);
- ✓ a formalized proof of correctness in *Lean* of the main derivation theorem of $RLTL\langle\mathcal{A}\rangle$ (Theorem 4) that links the incremental derivative based unfolding with the formal semantics.

Our combination allows regex *complement*, that is not supported in SPOT/PSL but can be supported naturally by using transition terms. We also lift the classical concept of ω -regular languages [Büchi 1960; McNaughton 1966] as the languages accepted by ABW so as to be modulo \mathcal{A} , and show that $RLTL\langle\mathcal{A}\rangle$ precisely captures ω -regularity modulo \mathcal{A} . Section 8 reviews related work and Section 9 discusses future work. Section 10 concludes the paper.

2 Preliminaries

As a meta-notation throughout the paper we write $lhs \stackrel{\text{DEF}}{=} rhs$ to let *lhs* be *equal by definition* to *rhs*. In the following let Σ be a nonempty, possibly infinite, *universe of core elements*. The universe of natural numbers is denoted by \mathbb{N} .

2.1 Effective Boolean Algebras

An *Effective Boolean Algebra (EBA)* over an element universe Σ [D'Antoni and Veanes 2021] is a tuple $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, ^c)$ where \mathcal{A} is a set of *predicates* that is closed under the Boolean connectives and contains \perp and \top . For $a \in \Sigma$ and $\alpha \in \mathcal{A}$ the *models* relation $a \models \alpha$ with $\llbracket \alpha \rrbracket \stackrel{\text{DEF}}{=} \{a \in \Sigma \mid a \models \alpha\}$ obeys classical Tarski laws such that $\llbracket \perp \rrbracket = \emptyset$ and $\llbracket \top \rrbracket = \Sigma$. For $\alpha, \beta \in \mathcal{A}$ let $\alpha \equiv \beta \stackrel{\text{DEF}}{=} \llbracket \alpha \rrbracket = \llbracket \beta \rrbracket$. If $\alpha \equiv \perp$ then α is *unsatisfiable* ($\text{UNSAT}(\alpha)$) else *satisfiable* ($\text{SAT}(\alpha)$). We require all the connectives to be *computable* and \models to be *decidable*.

Let $\Gamma \subseteq \mathcal{A}$ be finite. A *minterm* of Γ is a *satisfiable* predicate $(\bigwedge_{\alpha \in S} \alpha) \sqcap (\bigwedge_{\alpha \in \Gamma \setminus S} \alpha^c)$ for some subset S of Γ , where $(\bigwedge_{\alpha \in \emptyset} \dots) \stackrel{\text{DEF}}{=} \top$. Thus, all predicates in $\text{Minterms}(\Gamma)$ are satisfiable, mutually disjoint, and each satisfiable predicate in Γ is equivalent to a disjunction of some minterms. So $|\text{Minterms}(\Gamma)| \leq 2^{|\Gamma|}$. For example, $\text{Minterms}(\{\alpha, \beta\}) \subseteq \{\alpha \sqcap \beta, \alpha^c \sqcap \beta, \alpha \sqcap \beta^c, \alpha^c \sqcap \beta^c\}$.

We let \mathcal{A} also stand for the EBA itself and say that \mathcal{A} is *decidable* if $\text{SAT}(\alpha)$ is decidable. In most of the paper we use $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, ^c)$ as a given *alphabet EBA*.

2.2 Boolean Closures

Given a nonempty set S , we define the *Boolean closure* $\mathbb{B}(S)$ of S such that $S \subseteq \mathbb{B}(S)$ and $\mathbb{B}(S)$ is closed under \vee , \wedge , and \neg (as some given Boolean operators). We write $\mathbb{B}^+(S)$ for the *positive* Boolean closure where \neg is omitted. We associate a *weak equivalence* relation \cong over $\mathbb{B}(S)$ defining \wedge and \vee both as *associative*, *commutative*, and *idempotent* (ACI) operations. Weak equivalence is lifted to

transition terms with leaves in $\mathbb{B}(S)$ in Section 3 and plays a key role in maintaining *finiteness* of induced state space in later automata constructions.

The disjunctive normal form of $\phi \in \mathbb{B}^+(S)$ is a disjunction $\bigvee_{i=0}^n \psi_i$ where each ψ_i is a conjunction $\bigwedge X_i$ for some $X_i \subseteq S$. We adopt the *set-of-sets* representation of $\text{DNF}(\phi)$ as $\{X_i\}_{i=0}^n$. For example, $\text{DNF}((s_1 \vee s_2) \wedge s_3) = \{\{s_1, s_3\}, \{s_2, s_3\}\}$ that stands for $(s_1 \wedge s_3) \vee (s_2 \wedge s_3)$ and $\text{DNF}(s_1) = \{\{s_1\}\}$.

Let $\varphi = \text{DNF}(\phi)$ and $\psi = \text{DNF}(\psi)$. Then we can define \vee and \wedge directly over the set-of-sets representation by $\varphi \vee \psi \stackrel{\text{DEF}}{=} \varphi \cup \psi$ and $\varphi \wedge \psi \stackrel{\text{DEF}}{=} \{X \cup Y \mid X \in \varphi, Y \in \psi\}$. Observe that $\text{DNF}(\phi) \subseteq 2^S$ and in DNF, \emptyset acts as the unit element (\perp) of \vee , and $\{\emptyset\}$ acts as the unit element (\top) of \wedge . Intuitively, DNF is going to be used locally on small formulas. A key advantage here is that ACI of the *intended semantics* of the operations is built-in into the set-of-sets representation.

2.3 Words and Streams

A *stream* or *infinite word* $w \in \Sigma^\omega$ is a function from \mathbb{N} to Σ with $w_i \stackrel{\text{DEF}}{=} w(i)$. For *finite* words $v \in \Sigma^*$ the i 'th element v_i of v is defined for $0 \leq i < |v|$ where $|v|$ is the length of v . Concatenation of $v \in \Sigma^*$ with $w \in \Sigma^\omega$ (or $w \in \Sigma^*$) is the word or stream $v \cdot w$ (or vw) such that if $i < |v|$ then $(vw)_i$ is v_i else $w_{i-|v|}$. Let $w \in \Sigma^\omega$ and $n \in \mathbb{N}$. Let $\text{head}(w) \stackrel{\text{DEF}}{=} w_0$. We use the following stream operations that are adopted from the standard Stream library in Lean in order to align with the terminology used in the Lean formalization in Section 7.6.

$\text{drop}(n, w) \in \Sigma^\omega$ is the suffix of w s.t., for all i , $\text{drop}(n, w)_i = w_{n+i}$; $\text{tail}(w) \stackrel{\text{DEF}}{=} \text{drop}(1, w)$.

$\text{take}(n, w) \in \Sigma^*$ is the prefix of w of length n such that $\text{take}(n, w)_i = w_i$ for $i < n$.

If $a \in \Sigma$ then $a::w$ is the stream with head a and tail w , and a^ω is the stream $(a^\omega)_i = a$ for all i .

If $L \subseteq \Sigma^*$ then L^ω is the set of all $w \in \Sigma^\omega$ for which there exists $\Delta \in \mathbb{N}^\omega$ such that for all i , $\text{take}(1+\Delta_i, \text{drop}(\sum_{j<i}(1+\Delta_j), w)) \in L$ where $\sum_{j<0}(\dots) \stackrel{\text{DEF}}{=} 0$. Note that $L^\omega = (L \setminus \{\epsilon\})^\omega$. For example, $\{a\}^\omega = \{a^\omega\}$ where $\Delta = 0^\omega$.

We also use the following, more light-weight, math notations for stream operations. For $w \in \Sigma^\omega$ and $i \geq 0$, let $w_{..i} \stackrel{\text{DEF}}{=} \text{take}(i+1, w)$ and $w_{i..} \stackrel{\text{DEF}}{=} \text{drop}(i, w)$. Note that $w_{..0} = w_0$ and $w_{0..} = w$.

2.4 Languages and Derivatives

Let \mathfrak{D} be either Σ^ω or Σ^* , with the associated definition of complement $\mathfrak{C}(L) \stackrel{\text{DEF}}{=} \mathfrak{D} \setminus L$ for $L \subseteq \mathfrak{D}$. The *derivative* of L for $a \in \Sigma$ is $\mathbf{D}_a(L) \stackrel{\text{DEF}}{=} \{v \mid av \in L\}$. Irrespective of choice of \mathfrak{D} , one can show that $\mathbf{D}_a(\mathfrak{C}(L)) = \mathfrak{C}(\mathbf{D}_a(L))$ and $\mathbf{D}_a(L \cap L') = \mathbf{D}_a(L) \cap \mathbf{D}_a(L')$.

We use the language denotation function \mathcal{L} in relation to Σ^* and \mathcal{L} in relation to Σ^ω .

2.5 Extended Regular Expressions

Consider an EBA $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, \cdot^c)$. $\text{ERE}(\mathcal{A})$ or **ERE** for short is defined by the following abstract grammar, where $\alpha \in \mathcal{A}$. We let $R \in \text{ERE}$, R is called a *regex*. All operators are in order of precedence where \cup binds weakest. Let $R+ \stackrel{\text{DEF}}{=} R \cdot R^*$.

$$R ::= \alpha \mid \epsilon \mid R_1 \cup R_2 \mid R_1 \sqcap R_2 \mid R_1 \cdot R_2 \mid R^* \mid \sim R$$

Union (\cup), *intersection* (\sqcap), and *complement* (\sim), give rise to the EBA $(\Sigma^*, \text{ERE}, \models, \perp, \top, \cup, \sqcap, \sim)$ where $v \models R \Leftrightarrow v \in \mathcal{L}(R)$ with $\mathcal{L}(R)$ having the standard language semantics for **ERE** where, for all $a \in \Sigma$: $a \models \alpha \Leftrightarrow a \models \alpha$. **RE** denotes the *standard* fragment of **ERE** without \sqcap and \sim . For **ERE** we use *weak equivalence* \cong to simplify regexes in a light-weight manner so that language semantics is preserved: \cup and \sqcap are ACI, ϵ is the unit of \cdot , $\sim \perp \cong \top$, and further rules such as $\sim \epsilon \cong \top$.

The standard notation for the regex union operator is $|$, and $\&$ is typically used as the regex intersection operator. Here we use \cup and \sqcap to better align with the Lean formalization in Section 7.6 where the symbols $|$ and $\&$ are unavailable to this end.

3 Transition Terms

Let $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, \neg, \text{c})$ be a fixed EBA and let \mathcal{B} be a set or type called *leaves*. Here we provide a generalized view of *transition regexes* from [Stanford et al. 2021] that can be applied both to represent derivatives over Σ^* as well as Σ^ω . The set of *transition terms* $\mathbf{TTerm}(\mathcal{A}, \mathcal{B})$ (\mathbf{TTerm} for brevity) is defined by the following abstract grammar where $f \in \mathbf{TTerm}$, $\alpha \in \mathcal{A}$, and $\ell \in \mathcal{B}$:

$$f ::= \ell \quad | \quad (\alpha ? f_1 : f_2)$$

where α is called the *condition*, f_1 the *then-case* and f_2 the *else-case* of $(\alpha ? f_1 : f_2)$. While the formalization of \mathbf{TTerm} as an *inductive type* in Lean in Section 7.6 requires a constructor for leaves, we omit it here to avoid leaf conversion rules (much like $\Sigma \subseteq \Sigma^*$ while Σ^* is `List Σ` in Lean).

We work in a similar but more general setting than if-then-else terms in SMT and primarily want to avoid *lift rules* [Stanford et al. 2021, Section 4.1] that we consider as an implementation aspect of $\mathbf{TTerm}(\mathcal{A}, \mathcal{B})$ that depends on $\mathcal{B} = \mathbf{ERE}(\mathcal{A})$. All binary operations $\diamond : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}'$, and unary operations $\blacklozenge : \mathcal{B} \rightarrow \mathcal{B}'$ are, by definition, always *lifted* to \mathbf{TTerm} :

$$(\alpha ? f : g) \diamond h \stackrel{\text{DEF}}{=} (\alpha ? f \diamond h : g \diamond h) \quad \ell \diamond (\alpha ? f : g) \stackrel{\text{DEF}}{=} (\alpha ? \ell \diamond f : \ell \diamond g) \quad (1)$$

$$\blacklozenge(\alpha ? f : g) \stackrel{\text{DEF}}{=} (\alpha ? \blacklozenge f : \blacklozenge g) \quad (2)$$

When $\perp \in \mathcal{B}$ we use $(\alpha ? f) \stackrel{\text{DEF}}{=} (\alpha ? f : \perp)$ with \perp as the *implicit else-case*. Observe therefore that $(\alpha ? f) \diamond g = (\alpha ? f \diamond g : \perp \diamond g)$ and $\blacklozenge(\alpha ? f) = (\alpha ? \blacklozenge f : \blacklozenge \perp)$.

The case when \mathcal{B} and \mathcal{B}' are different types is needed in Sections 5 and 7.3. Given $a \in \Sigma$ and $f \in \mathbf{TTerm}(\mathcal{A}, \mathcal{B})$, the *evaluation (leaf)* of f for a , denoted by $f[a]$, is defined as follows.

$$\ell[a] \stackrel{\text{DEF}}{=} \ell, \quad (\alpha ? f_1 : f_2)[a] \stackrel{\text{DEF}}{=} \text{if } a \models \alpha \text{ then } f_1[a] \text{ else } f_2[a]$$

Therefore, the following facts hold for all the lifted operations:

$$(f \diamond g)[a] = f[a] \diamond g[a] \wedge (\blacklozenge f)[a] = \blacklozenge(f[a]) \quad (3)$$

Weak equivalence of $f, g \in \mathbf{TTerm}(\mathcal{A}, \mathcal{B})$ is defined by $f \cong g \stackrel{\text{DEF}}{=} \forall a \in \Sigma : f[a] \cong g[a]$.

Leaves of f , $\text{lvs}(f)$, and conditions of f , $\text{conds}(f)$, are defined by induction over \mathbf{TTerm} .

A transition term f is *clean* when all of its branches are feasible, in particular $\forall \ell \in \text{lvs}(f) : \exists a : f[a] = \ell$. Many rewrites that preserve \cong can be applied incrementally during operations on \mathbf{TTerm} by incorporating *satisfiability* checking of \mathcal{A} to eliminate unreachable subterms. Some rewrites, such as $(_ ? f : f) \cong f$, $(\top ? f : _) \cong f$, and $(\perp ? _ : f) \cong f$ are always applied implicitly as trivial condition eliminations.

Example 3.1. Let φ belong to \mathcal{B} and let α and β belong to \mathcal{A} such that α implies β . The rewrites below illustrate a typical scenario where the initial transition term (the one in the first row below) is computed and simplified. Let \mathcal{B} be an EBA with disjunction \vee , complement \neg , bottom element \perp , and top element \top (to distinguish it from \top in \mathcal{A}). The law of excluded middle in \mathcal{B} is thus $\neg\varphi \vee \varphi \cong \top$, and further weak equivalence laws include $\neg\perp \cong \top$ and $\top \vee \varphi \cong \top$.

$$\begin{array}{c} \neg(\alpha ? \varphi : \perp) \vee (\beta ? \varphi : \perp) \\ \hline (\alpha ? (\beta ? \neg\varphi \vee \varphi : \neg\varphi \vee \perp) : (\beta ? \neg\perp \vee \varphi : \neg\perp \vee \perp)) \quad (\text{lift } \neg \text{ and } \vee \text{ to } \mathbf{TTerm}(\mathcal{A}, \mathcal{B})) \\ \hline (\alpha ? (\beta ? \top : \neg\varphi) : (\beta ? \top : \perp)) \quad (\text{rewrites of leaves based on } \cong \text{ in } \mathcal{B}) \\ \hline (\alpha ? (\beta ? \top : \neg\varphi) : \perp) \quad (\text{trivial condition elimination of } \beta) \\ \hline (\alpha ? \perp : \perp) \quad (\text{UNSAT}(\alpha \sqcap \beta^c) \text{ in } \mathcal{A} \text{ so } (\beta ? \top : \neg\varphi) \text{ rewrites to } \perp) \\ \hline \perp \quad (\text{trivial condition elimination of } \alpha) \end{array}$$

More generally, $\mathbf{TTerm}(\mathcal{A}, \mathcal{B})$ becomes a derived EBA with its own set of rewrite rules. Elimination of unreachable subterms, as with $\neg\varphi$ above, is called *cleaning* in [Stanford et al. 2021]. \square

When \mathcal{B} is an EBA modulo \mathcal{A} , such as $\text{ERE}(\mathcal{A})$, the *implementation* of $\text{TTerm}(\mathcal{A}, \mathcal{B})$ can maintain a symbolic representation of the lifted Boolean operators of \mathcal{B} . In particular, it can be beneficial not to always propagate disjunctions into leaves by (1), but to maintain top-level disjunctions as operators between transition terms.

Example 3.2. Consider $\text{TTerm}(\mathcal{A}, \text{ERE}(\mathcal{A}))$. A typical case is that the symbolic derivative of a regex $\alpha \cdot L \cup \beta \cdot R$, where α and β are predicates in \mathcal{A} , has the form $(\alpha ? L) \cup (\beta ? R)$. Rather than always computing $(\alpha ? (\beta ? L \cup R : L) : (\beta ? R))$ the union operator \cup is maintained as a symbolic operator between the transition regexes as $(\alpha ? L) \cup (\beta ? R)$. We can call it the *Antimirov normal form* in this case, in analogy with [Antimirov 1996]. Similar representation can be maintained for $\text{RLTL}(\mathcal{A})$. One key advantage is to avoid unnecessary cleaning when it is irrelevant how the conditions in the separate disjuncts relate to each other (e.g., whether α and β overlap or not). \square

4 Alternating Büchi Word Automata Modulo Theories

We now turn our attention to alternating Büchi automata over words or *ABW* (which admit a linear time translation from LTL [Tsai and Vardi 2021]) and show how they can be generalized using transition terms so as to be modulo $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, \cdot^c)$. The key aspect of the following definition is that transitions in the automata are represented symbolically by transition terms.

An *alternating Büchi automaton modulo \mathcal{A}* ($ABW_{\mathcal{A}}$) is a tuple $M = (\mathcal{A}, Q, q^0, \varrho, F)$ where Q is a finite set of states, $q^0 \in \mathbb{B}^+(Q)$ is an initial state combination, $F \subseteq Q$ is a set of accepting states, and $\varrho : Q \rightarrow \text{TTerm}(\mathcal{A}, \mathbb{B}^+(Q))$ is a transition function.

Two $ABW_{\mathcal{A}}$'s M and N are *compatible* if all shared states of M and N have identical behaviors:

$$\forall q \in Q_M \cap Q_N : \varrho_M(q) = \varrho_N(q) \wedge (q \in F_M \Leftrightarrow q \in F_N)$$

Boolean operations $\diamond \in \{\wedge, \vee\}$ over compatible $M, N \in ABW_{\mathcal{A}}$, are:

$$M \diamond N \stackrel{\text{DEF}}{=} (\mathcal{A}, Q_M \cup Q_N, q_M^0 \diamond q_N^0, \varrho_M \cup \varrho_N, F_M \cup F_N)$$

In figures we show transition terms $\varrho(q)$ by *symbolic transitions* $q \xrightarrow{\beta} \phi$ where $\phi \in \mathbb{B}^+(Q)$ is a leaf of $\varrho(q)$ guarded by the branch condition $\beta \in \mathcal{A}$ from the root of $\varrho(q)$ to ϕ . The automaton M (or ϱ_M) is *clean* when all the transition terms in ϱ_M are clean. For example, the automaton in Figure 1 is clean when both α and α^c are satisfiable.

The *DNF normal form* of ϱ , $\text{DNF}(\varrho)$, maps $q \in Q$ to $\text{DNF}(\varrho(q))$. Let $\varrho = \text{DNF}(\varrho)$. We let also $\text{conds}(M) \stackrel{\text{DEF}}{=} \bigcup_{q \in Q} \text{conds}(\varrho(q))$ denote the set of all conditions in ϱ . E.g., in Figure 1, $\text{conds}(M) = \{\alpha\}$ and $\varrho(q_0) = (\alpha ? \{\{q_2, q_0\}\} : \{\{q_1, q_0\}\})$.

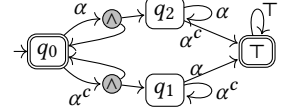


Fig. 1. $ABW_{\mathcal{A}}$.

4.1 Runs and Languages

Let $M = (\mathcal{A}, Q, q^0, \varrho, F)$ be an $ABW_{\mathcal{A}}$. The standard definition of runs [Vardi 1995] uses $X \subseteq Q$ to specify *satisfying assignments* for $\phi \in \mathbb{B}^+(Q)$ as follows where \models is standard (Tarski) semantics:

$$X \text{ satisfies } \phi \stackrel{\text{DEF}}{=} \bigcup_{q \in X} \{q \mapsto \text{true}\} \cup \bigcup_{q \in Q \setminus X} \{q \mapsto \text{false}\} \models \phi$$

LEMMA 1. $\forall \phi \in \mathbb{B}^+(Q), X \subseteq Q : X \text{ satisfies } \phi \Leftrightarrow \exists Y \subseteq X : Y \in \text{DNF}(\phi)$.

PROOF. We use that X satisfies ϕ iff some $X_{\min} \subseteq X$ satisfies ϕ in a minimal manner [Kupferman 2018, 4.5.1], that we can formally define here by using $\text{DNF}(\phi)$ (recall that $\text{DNF}(\phi) \subseteq 2^Q$):

$$X_{\min} \in \text{antichain}_{\subseteq}(\text{DNF}(\phi)) \quad \text{antichain}_{\subseteq}(X) \stackrel{\text{DEF}}{=} \{x \in X \mid \nexists y \in X : y \subsetneq x\} \quad (4)$$

Note that $\text{antichain}_{\subseteq}(\text{DNF}(\phi)) \subseteq \text{DNF}(\phi)$ and if $Y \in \text{DNF}(\phi)$ then Y satisfies ϕ since $\phi \in \mathbb{B}^+(Q)$. \square

Let $aw \in \Sigma^\omega$ and $q \in Q$. A *run for aw from q* is a (potentially infinite) Q -labelled tree τ whose root node has label q and, by using Lemma 1, there exists $X \in \text{DNF}(\varrho(q)[a])$ such that, for each $p \in X$, there is an immediate subtree τ_p of τ such that τ_p is a run for w from p .

A run τ is *accepting* if *all* of the infinite branches of τ visit F infinitely often, where an *infinite branch* visits the root of τ and then continues as an infinite branch from some immediate subtree of τ . For $w \in \Sigma^\omega$ and $q \in Q$: w is *accepted from q* iff there exists an accepting run for w from q ; the *language of q in M* is defined and lifted to $\mathbb{B}^+(Q)$ as follows:

$$\begin{aligned} \mathcal{L}_M(q) &\stackrel{\text{DEF}}{=} \{w \in \Sigma^\omega \mid w \text{ is accepted from } q\} \\ \mathcal{L}_M(\mathbf{q} \vee \mathbf{p}) &\stackrel{\text{DEF}}{=} \mathcal{L}_M(\mathbf{q}) \cup \mathcal{L}_M(\mathbf{p}) \quad \mathcal{L}_M(\mathbf{q} \wedge \mathbf{p}) \stackrel{\text{DEF}}{=} \mathcal{L}_M(\mathbf{q}) \cap \mathcal{L}_M(\mathbf{p}) \quad \mathcal{L}(M) \stackrel{\text{DEF}}{=} \mathcal{L}_M(\mathbf{q}^0) \end{aligned} \quad (5)$$

For the positive Boolean operations it follows that $\mathcal{L}(M \wedge N) = \mathcal{L}(M) \cap \mathcal{L}(N)$ and $\mathcal{L}(M \vee N) = \mathcal{L}(M) \cup \mathcal{L}(N)$, as in the classical case [Kupferman 2018, Theorem 20]. Complementation of an $ABW_{\mathcal{A}}$ M is theoretically possible via alternation elimination (Section 5) and mintermization (Section 4.1.1) followed by [Kupferman 2018, Theorem 4], but the existence of a *symbolic* algorithm that would construct an $ABW_{\mathcal{A}}$ $\neg M$ from M such that $\mathcal{L}(\neg M) = \mathbb{C}(\mathcal{L}(M))$ is an open problem.

We adopt the same notation also for classical ABW M that $\mathcal{L}_M(\mathbf{q})$ denotes the language of M accepted starting from the state combination $\mathbf{q} \in \mathbb{B}^+(Q_M)$.

4.1.1 From $ABW_{\mathcal{A}}$ to Classical ABW . A variety of decidability results for $ABW_{\mathcal{A}}$ follow directly by reduction to classical ABW (this does not imply a practical implementation scheme, as it requires mintermization). Let $A = \text{Minterms}(\text{conds}(M))$ be a finite alphabet and for $a \in \Sigma$ let $\hat{a} \in A$ denote the minterm such that $a \models \hat{a}$. Let \hat{M} denote the classical $ABW (A, Q, \mathbf{q}^0, \hat{\varrho}, F)$ where $\hat{\varrho}(q, \hat{a}) \stackrel{\text{DEF}}{=} \varrho(q)[a]$ and lift $\hat{a} \in A$ to $\hat{w} \in A^\omega$. Then

LEMMA 2. $\forall M \in ABW_{\mathcal{A}}, \phi \in \mathbb{B}^+(Q_M), w \in \Sigma^\omega : \hat{w} \in \mathcal{L}_{\hat{M}}(\phi) \Leftrightarrow w \in \mathcal{L}_M(\phi)$.

In particular, *decidability* of $\mathcal{L}(M) \neq \emptyset$, provided that \mathcal{A} is decidable, follows from decidability of $\mathcal{L}(\hat{M}) \neq \emptyset$ [Miyano and Hayashi 1984; Rabin 1970].

4.1.2 From Classical ABW to $ABW_{\mathcal{A}}$. In the opposite direction, consider a classical $ABW M = (2^P, Q, \mathbf{q}^0, \rho, F)$ where P is a finite set of propositions. For all $a \subseteq P$ let $\alpha_a \stackrel{\text{DEF}}{=} (\bigwedge_{p \in a} p) \sqcap (\bigwedge_{p \in P \setminus a} p^c)$. M is lifted to $M_{\mathcal{A}} = (\mathcal{A}, Q, \mathbf{q}^0, \varrho, F)$ where the EBA for \mathcal{A} is essentially a SAT solver for $\mathbb{B}(P)$ and ϱ maps $q \in Q$ to $\bigvee_{a \subseteq P} (\alpha_a ? \rho(q, a))$. Then $\varrho(q)[a] = \rho(q, a)$ and so $\mathcal{L}(M) = \mathcal{L}(M_{\mathcal{A}})$.

4.2 Nondeterministic Büchi Word Automata Modulo Theories

An $ABW_{\mathcal{A}}$ M is *nondeterministic*, an $NBW_{\mathcal{A}}$, if all leaves of all transition terms in ϱ_M and \mathbf{q}_M^0 are disjunctions of states. One can view an $NBW_{\mathcal{A}}$ equivalently as a finite automaton modulo \mathcal{A} [D'Antoni and Veanes 2021] but with the Büchi acceptance condition, i.e., that each transition term $\varrho_M(q)$ for $q \in Q_M$ has the equivalent representation as $\bigvee_{i=1}^n (\alpha_i ? q_i)$ where $q_i \in Q_M$.

Decidability of nonemptiness of a clean $NBW_{\mathcal{A}}$ has *linear time* complexity, that follows by adapting the corresponding result from [Emerson and Lei 1985, 1987], where cleaning, as a separate preprocessing step, has computational complexity that depends on that of satisfiability in \mathcal{A} but is otherwise linear in the size of M . In particular, mintermization, i.e., conversion to the classical setting, is avoided.

NBW s play a prominent role in model-checking. Of particular interest is *product* of NBW s that is used to determine if a specification of bad behavior, translated into an NBW , intersects with a system model that is also given as an NBW . The ability to define and work with such models *modulo background theories*, i.e., with $NBW_{\mathcal{A}}$ directly, can increase their succinctness by an *exponential* factor without later on incurring that cost in algorithms, as shown in Section 5.3.

The alternation elimination algorithm \mathcal{A} in Section 5 is used to define a product algorithm for $NBW_{\mathcal{A}}$'s in Section 5.3 that is *quadratic* in the size of the automata provided that satisfiability checking in \mathcal{A} has constant cost. Most importantly, mintermization is avoided.

5 Alternation Elimination

Although alternating Büchi automata are an attractive target for temporal logic, incurring no space blowup, the problem of testing for nonemptiness is harder for alternating automata (due to the presence of both \wedge and \vee in the state formula) than non-alternating automata [Boker et al. 2010].

Let $M = (\mathcal{A}, Q, q^0, \varrho, F)$ be an $ABW_{\mathcal{A}}$ modulo $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, \neg, \neg^c)$. We introduce an alternation elimination algorithm for M that is a derivative-based generalization of the algorithm in [Miyano and Hayashi 1984] (see also [Vardi 1995, Proposition 20]). The algorithm constructs an $NBW_{\mathcal{A}}$ $N = (\mathcal{A}, S, S_0, \sigma, F_N)$ accepting the same language as M . As shown by [Boker et al. 2010], the concept behind [Miyano and Hayashi 1984] accurately captures the prime concern in alternation removal, which is the need to associate the states of the equivalent $NBW_{\mathcal{A}}$ N with two sets of states from M , which means that a $\Omega(3^{|Q|})$ space blowup cannot be avoided. Let $\varrho = \text{DNF}(\varrho)$.

In the rest of this section let $Q = 2^{2^Q}$, $P = 2^Q \times 2^Q$, and $\mathcal{P} = 2^P$. Q is the leaf type of transition terms of M in DNF, i.e., $\varrho : Q \rightarrow \mathbf{TTerm}(\mathcal{A}, Q)$. \mathcal{P} is the leaf type of transition terms of N where P is the *state type* of N , i.e., $\sigma : S \rightarrow \mathbf{TTerm}(\mathcal{A}, \mathcal{P})$. $F_N = \{\langle U, V \rangle \in S \mid U = \emptyset\}$.

5.1 Algorithm \mathcal{A}

We start by defining the key operation $\@$ called *alternation product* or *alternation times* of $f, g \in \mathbf{TTerm}(\mathcal{A}, Q)$ as a term $f \@ g \in \mathbf{TTerm}(\mathcal{A}, \mathcal{P})$. Let $\varphi, \psi \in Q$ and first define $\@ : Q \times Q \rightarrow \mathcal{P}$:

$$\varphi \@ \psi \stackrel{\text{DEF}}{=} \{\langle X \setminus F, Y \cup (X \cap F) \rangle \mid X \in \varphi, Y \in \psi\} \quad (6)$$

The purpose of $\@$ is to partition the states into those that have not yet visited a final state ($X \setminus F$) and those that have ($Y \cup (X \cap F)$). Observe that $\@$ is not symmetric and $\{\emptyset\}$ intuitively denotes \top in Q . For example, if $q \notin F$ then $\{\{q\}\} \@ \{\emptyset\} = \{\langle q, \emptyset \rangle\}$ is not a final state of N while $\{\emptyset\} \@ \{\{q\}\} = \langle \emptyset, \{q\} \rangle$ is a final state of N . Recall that (1) lifts $\@$ to $\mathbf{TTerm}(\mathcal{A}, Q) \times \mathbf{TTerm}(\mathcal{A}, Q) \rightarrow \mathbf{TTerm}(\mathcal{A}, \mathcal{P})$.

For $h \in \mathbf{TTerm}(\mathcal{A}, \mathcal{P})$ let $\text{states}(h) \stackrel{\text{DEF}}{=} \bigcup \text{Ivs}(h)$. So $\text{states}(h) \subseteq P$. The algorithm \mathcal{A} constructs an equivalent $NBW_{\mathcal{A}}$ from M . For $U, V \subseteq Q$ let

$$\begin{aligned} \varrho(U) &\stackrel{\text{DEF}}{=} \begin{cases} \{\emptyset\}, & \text{if } U = \emptyset; \\ \bigwedge_{q \in U} \varrho(q), & \text{otherwise.} \end{cases} \\ \@_{\varrho}(\langle U, V \rangle) &\stackrel{\text{DEF}}{=} \begin{cases} \varrho(V) \@ \{\emptyset\}, & \text{if } U = \emptyset; \\ \varrho(U) \@ \varrho(V), & \text{otherwise.} \end{cases} \end{aligned}$$

and let $\text{dom}(\sigma)$ denote the domain of the partially defined transition function σ in the while loop of the \mathcal{A} algorithm that computes the fixpoint of S :

$$\mathcal{A}(M) \stackrel{\text{DEF}}{=} \begin{cases} S_0 = \text{DNF}(q^0) \@ \{\emptyset\}; S \leftarrow S_0; \sigma \leftarrow \emptyset; \\ \textbf{while } \exists s \in S \setminus \text{dom}(\sigma) \textbf{ let } f = \@_{\varrho}(s) \textbf{ in } \sigma(s) \leftarrow f; S \leftarrow S \cup \text{states}(f) \\ \textbf{return } (\mathcal{A}, S, S_0, \sigma, \{\langle U, V \rangle \in S \mid U = \emptyset\}) \end{cases}$$

The key power of the algorithm lies in its modular use of \mathbf{TTerm} that enables eager simplification of transition terms $f \@ g$ by utilizing satisfiability checking in \mathcal{A} , both by cleaning and by rules like Lemma 3 below. That said, simplifications in \mathbf{TTerm} , while essential for implementing \mathcal{A} , require satisfiability in \mathcal{A} to be *decidable*, which is not necessary for the formal correctness (Theorem 1).

We make use of the following *state reduction* lemma in the following example, where the number of states would otherwise *double*.

LEMMA 3 (STATE REDUCTION). *If $U' \subseteq U$, $V' \subseteq V$, and $U = \emptyset \Leftrightarrow U' = \emptyset$ then if $\varrho(U') = \varrho(U)$ and $\varrho(V') = \varrho(V)$ then $\langle U, V \rangle$ can be replaced by $\langle U', V' \rangle$ in the \mathcal{A} algorithm.*

PROOF. The computation of f preserves language equivalence if we identify states with identical transition terms and acceptance conditions. \square

Example 5.1. Let M be as in Figure 1, where $q^0 = q_0$, $F = \{q_0, \top\}$, and

$\varrho = \{q_0 \mapsto (\alpha ? q_2 \wedge q_0 : q_1 \wedge q_0), q_1 \mapsto (\alpha ? \top : q_1), q_2 \mapsto (\alpha ? q_2 : \top), \top \mapsto \top\}$

Each row of the following table represents one iteration of the while-loop in the run of $\mathcal{A}(M)$. Since $q_0 \in F$, observe that $\text{DNF}(q_0) @ \{\emptyset\} = \{\{q_0\}\} @ \{\emptyset\} = \{\langle \emptyset, \{q_0\} \rangle\}$.

s	$\sigma(s)$	Resulting NBW $_{\mathcal{A}}$
$s_0 = \langle \emptyset, \{q_0\} \rangle$	$(\alpha ? \{\{q_2, q_0\}\} : \{\{q_1, q_0\}\}) @ \{\emptyset\}$ $\cong (\alpha ? \{\langle \{q_2\}, \{q_0\} \rangle\} : \{\langle \{q_1\}, \{q_0\} \rangle\})$	
$s_1 = \langle \{q_1\}, \{q_0\} \rangle$	$\varrho(q_1) @ \varrho(q_0)$ $\cong (\alpha ? \{\langle \emptyset, \{q_2, q_0\} \rangle\} : \{\langle \{q_1\}, \{q_1, q_0\} \rangle\})$ (Lma 3) $\cong (\alpha ? \{\langle \emptyset, \{q_0\} \rangle\} : \{\langle \{q_1\}, \{q_0\} \rangle\})$	
$s_2 = \langle \{q_2\}, \{q_0\} \rangle$	(Lma 3) $\cong (\alpha ? \{\langle \{q_2\}, \{q_0\} \rangle\} : \{\langle \emptyset, \{q_0\} \rangle\})$	

The constructions of $\sigma(s_1)$ and $\sigma(s_2)$ applied Lemma 3 whereby both $\{q_1, q_0\}$ and $\{q_2, q_0\}$ simplify to $\{q_0\}$. Cleaning was used to eliminate nested ITEs with condition α . \square

5.2 Correctness of \mathcal{A}

The correctness proof of the algorithm is by reduction to [Miyano and Hayashi 1984], using generic properties of transition terms and mintermization. Given a pair $\langle U, V \rangle$ of finite sets U, V of states let $\wedge \langle U, V \rangle \stackrel{\text{DEF}}{=} \wedge (U \cup V)$.

THEOREM 1 (\mathcal{A}). $\forall q \in Q_{\mathcal{A}(M)} : \mathcal{L}_{\mathcal{A}(M)}(q) = \mathcal{L}_M(\wedge q)$ and $\mathcal{L}(\mathcal{A}(M)) = \mathcal{L}(M)$.

PROOF. Let $N = \mathcal{A}(M)$ as above. The algorithm terminates because P is finite and $S \subseteq P$. First let $q = \langle U, V \rangle$ be such that $U \neq \emptyset$. Note that $V = \emptyset$ implies that $\varrho(V) = \{\emptyset\}$.

$$\begin{aligned}
 \sigma(\langle U, V \rangle)[a] &= (\varrho(U) @ \varrho(V))[a] \\
 &= (\text{DNF}(\wedge_{t \in U} \varrho(t)) @ \text{DNF}(\wedge_{t \in V} \varrho(t)))[a] \stackrel{(3)}{=} \text{DNF}(\wedge_{t \in U} (\varrho(t)[a])) @ \text{DNF}(\wedge_{t \in V} (\varrho(t)[a])) \\
 &\stackrel{(6)}{=} \{ \langle X \setminus F, Y \cup (X \cap F) \rangle \mid X \in \text{DNF}(\wedge_{t \in U} (\varrho(t)[a])), Y \in \text{DNF}(\wedge_{t \in V} (\varrho(t)[a])) \} \quad (7)
 \end{aligned}$$

Now let $q = \langle U, V \rangle$ be such that $U = \emptyset$. We get that

$$\begin{aligned}
 \sigma(\langle U, V \rangle)[a] &= (\varrho(V) @ \{\emptyset\})[a] = (\text{DNF}(\wedge_{t \in V} \varrho(t)) @ \{\emptyset\})[a] \stackrel{(3)}{=} \text{DNF}(\wedge_{t \in V} (\varrho(t)[a])) @ \{\emptyset\} \\
 &\stackrel{(6)}{=} \{ \langle X \setminus F, X \cap F \rangle \mid X \in \text{DNF}(\wedge_{t \in V} (\varrho(t)[a])) \} \quad (8)
 \end{aligned}$$

Next, we relate (7) and (8) with \widehat{M} (recall Section 4.1), where for $W \subseteq Q$, $\wedge_{t \in W} (\varrho(t)[a]) = \wedge_{t \in W} \widehat{\varrho}(t, \widehat{a})$ – (7) and (8) represent the construction used in [Miyano and Hayashi 1984] (MH84) on top of which further analysis over accepting runs of \widehat{M} is built, and where choosing all satisfiers of ϕ from $\text{DNF}(\phi)$ is sufficient by Lemma 1. It follows, for all $s \in S$ and $w \in \Sigma^\omega$, that $w \in \mathcal{L}_N(s) \Leftrightarrow$ (by Lemma 2) $\widehat{w} \in \mathcal{L}_{\widehat{N}}(s) \Leftrightarrow$ (by MH84) $\widehat{w} \in \mathcal{L}_{\widehat{M}}(\wedge s) \Leftrightarrow$ (by Lemma 2) $w \in \mathcal{L}_M(\wedge s)$. So $\mathcal{L}(M) = \mathcal{L}_M(q^0) = \bigcup_{X \in \text{DNF}(q^0)} \mathcal{L}_M(\wedge X) = \bigcup_{s \in S_0} \mathcal{L}_M(\wedge s) = \bigcup_{s \in S_0} \mathcal{L}_N(s) = \mathcal{L}(N)$. \square

5.3 Product of NBW Modulo Theories

A direct application of \mathcal{AE} is the *product* of two compatible $NBW_{\mathcal{A}}$'s $N_i = (\mathcal{A}, Q_i, Q_i^0, \rho_i, F_i)$, for $i \in \{1, 2\}$ as the $NBW_{\mathcal{A}}$:

$$N_1 \times N_2 \stackrel{\text{DEF}}{=} \mathcal{AE}(N_1 \wedge N_2)$$

Product provides an essential step in the automata-based approach to model checking, as discussed in the introduction. We get the following corollary of Theorem 1. Assume, w.l.o.g., that $Q_1 \cap Q_2 = \emptyset$.

COROLLARY 1 (PRODUCT). $|Q_{N_1 \times N_2}| \leq 4|Q_1||Q_2| \wedge \forall \langle U, V \rangle \in Q_{N_1 \times N_2} :$

$$(1) U \cap V = \emptyset \wedge U \cap (F_1 \cup F_2) = \emptyset$$

$$(2) \exists q \in Q_1, p \in Q_2 : U \cup V = \{q, p\} \wedge \mathcal{L}_{N_1 \times N_2}(\langle U, V \rangle) = \mathcal{L}_{N_1}(q) \cap \mathcal{L}_{N_2}(p).$$

Recall that, in an $NBW_{\mathcal{A}}$, $\rho(q)$ is maintained as a disjunction of symbolic transitions representing the transition term $\bigvee_{i \in I} (\alpha_i ? \mathbf{q}_i)$ where $\mathbf{q}_i = \{\{q_i\}\}$ for some $q_i \in Q$. (Recall that $(\alpha_i ? \mathbf{q}_i)$ stands for $(\alpha_i ? \mathbf{q}_i : \emptyset)$ here, where \emptyset acts as \perp in DNF.) The alternation product $\rho(q) @ \rho(p)$ can then be restated more compactly as:

$$\bigvee_{i \in I} (\alpha_i ? \mathbf{q}_i) @ \bigvee_{j \in J} (\beta_j ? \mathbf{p}_j) \stackrel{\text{DEF}}{=} \bigvee_{i \in I, j \in J, \text{SAT}(\alpha_i \cap \beta_j)} (\alpha_i \cap \beta_j ? \mathbf{q}_i @ \mathbf{p}_j)$$

where *cleaning* ($\text{SAT}(\alpha_i \cap \beta_j)$) is embedded. Observe that $\emptyset @ \mathbf{p} = \mathbf{q} @ \emptyset = \emptyset$, so

$$(\alpha ? \mathbf{q}) @ (\beta ? \mathbf{p}) = (\alpha ? (\beta ? \mathbf{q} @ \mathbf{p} : \mathbf{q} @ \emptyset) : (\beta ? \emptyset @ \mathbf{p} : \emptyset @ \emptyset)) \cong (\alpha ? (\beta ? \mathbf{q} @ \mathbf{p})) \cong (\alpha \cap \beta ? \mathbf{q} @ \mathbf{p})$$

This means that, also the number of satisfiability tests is $O(|N_1||N_2|)$, that eliminate unreachable states. *In stark contrast, a translation to classical NBWs in $O(2^{|N_1|+|N_2|})$ due to the mintermization or bitblasting needed to establish the finite alphabet.*

6 LTL Modulo Theories

Here we lift classical linear temporal logic (LTL) to be modulo $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, \Box, \Box^c)$, or $\text{LTL}\langle \mathcal{A} \rangle$, via symbolic derivatives that translate $\text{LTL}\langle \mathcal{A} \rangle$ formulas into transition terms. We then utilize the algorithms developed for $ABW_{\mathcal{A}}$ for decision procedures of $\text{LTL}\langle \mathcal{A} \rangle$. Let $\alpha \in \mathcal{A}$ and $\varphi \in \text{LTL}\langle \mathcal{A} \rangle$.

$$\varphi ::= \alpha \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{X}\varphi \mid \varphi_1 \mathbf{U} \varphi_2 \mid \varphi_1 \mathbf{R} \varphi_2$$

where \mathbf{X} (Next), \mathbf{U} (Until), and \mathbf{R} (Release), are *modal operators*. The *true* formula is \top and the *false* formula is \perp from \mathcal{A} . The following standard abbreviations are also used: $\varphi \rightarrow \psi \stackrel{\text{DEF}}{=} \neg\varphi \vee \psi$, $\mathbf{F}\psi \stackrel{\text{DEF}}{=} \top \mathbf{U} \psi$, and $\mathbf{G}\psi \stackrel{\text{DEF}}{=} \perp \mathbf{R} \psi$, where \mathbf{F} is *Finally* (Eventually) and \mathbf{G} is *Globally* (Always).

6.1 Semantics

A stream $w \in \Sigma^\omega$ is a *model* of $\varphi \in \text{LTL}\langle \mathcal{A} \rangle$, $w \models \varphi$, when the following holds, where $\alpha \in \mathcal{A}$:

$$w \models \alpha \stackrel{\text{DEF}}{=} w_0 \models \alpha \tag{9}$$

$$w \models \varphi \wedge \psi \stackrel{\text{DEF}}{=} w \models \varphi \wedge w \models \psi \quad w \models \varphi \vee \psi \stackrel{\text{DEF}}{=} w \models \varphi \vee w \models \psi \quad w \models \neg\varphi \stackrel{\text{DEF}}{=} w \not\models \varphi \tag{10}$$

$$w \models \mathbf{X}\psi \stackrel{\text{DEF}}{=} \text{tail}(w) \models \psi \tag{11}$$

$$w \models \varphi \mathbf{U} \psi \stackrel{\text{DEF}}{=} \exists j : \text{drop}(j, w) \models \psi \wedge \forall i < j : \text{drop}(i, w) \models \varphi \tag{12}$$

$$w \models \varphi \mathbf{R} \psi \stackrel{\text{DEF}}{=} \forall j : \text{drop}(j, w) \models \psi \vee \exists j : \text{drop}(j, w) \models \varphi \wedge \forall i \leq j : \text{drop}(i, w) \models \psi \tag{13}$$

$$\mathcal{L}(\varphi) \stackrel{\text{DEF}}{=} \{w \in \Sigma^\omega \mid w \models \varphi\} \quad \varphi \equiv \psi \stackrel{\text{DEF}}{=} \mathcal{L}(\varphi) = \mathcal{L}(\psi) \tag{14}$$

The rules (12) and (13) are duals of each other, either one suffices as the main definition. Observe that if $\alpha \in \mathcal{A}$ then $w \models \neg\alpha \Leftrightarrow w \not\models \alpha \Leftrightarrow w_0 \not\models \alpha \Leftrightarrow w_0 \models \alpha^c \Leftrightarrow w \models \alpha^c$.

The following examples illustrate some cases of $\text{LTL}\langle\mathcal{A}\rangle$ modulo various \mathcal{A} . Example 6.1 illustrates – at a very abstract level – the well-known connection of integrating SAT solving into symbolic LTL, e.g., by using BDDs [Wulf et al. 2008].

Example 6.1. Classical LTL over a set of atomic propositions P is $\text{LTL}\langle\mathcal{A}\rangle$ where \mathcal{A} is an algebra of Boolean combinations over P , where $\Sigma = 2^P$ and $\mathcal{A} = \mathbb{B}(P)$, and where satisfiability can be implemented using a SAT solver. An element $d \in \Sigma$ such that $d \models \alpha$ defines a *truth assignment* to P that makes α true. For example, if $P = \{p_i\}_{i < 7}$ and $\alpha = p_6 \sqcap p_5 \sqcap p_4 \sqcap (p_3 \sqcup ((p_2^c) \sqcap p_1))$ then if $w \in \Sigma^\omega$ is such that $w_0 = \{p_1, p_4, p_5, p_6\}$ and $w_1 = \{p_1, p_2, p_4, p_5, p_6\}$ then $w \models \alpha$ but $\text{tail}(w) \not\models \alpha$. Thus, for example, $w \not\models \mathbf{G}\alpha$. \square

Unlike in Example 6.1, in Example 6.2 Σ is infinite.

Example 6.2. Consider LTL modulo the EBA $(\Sigma, \mathcal{A}, \models, \perp, \top, \vee, \wedge, \neg)$ of SMT formulas \mathcal{A} that can be implemented using an SMT solver. Here Σ is the set of interpretations to *uninterpreted constants* or *variables*. Let x be a variable of type *real*. Then $(x < 1) \mathbf{R} (0 < x)$ states that x must remain positive until $x < 1$. E.g., $(x \mapsto 1)(x \mapsto \frac{1}{2})(x \mapsto 0)^\omega \models (x < 1) \mathbf{R} (0 < x)$. \square

6.2 Vardi Derivatives of LTL Modulo Theories

Here we show how the semantics of $\text{LTL}\langle\mathcal{A}\rangle$, or LTL for short, can be realized via transition terms. The key observation here is that a concrete derivative (i.e., for a given symbol $a \in \Sigma$) is not actually constructed but maintained in a symbolic form as a transition term. Let $\alpha \in \mathcal{A}$, and $\varphi, \psi \in \text{LTL}$. The (*symbolic*) *derivative* of an LTL formula is defined as a term in $\mathbf{TTerm}\langle\mathcal{A}, \text{LTL}\rangle$:

$$\partial(\alpha) \stackrel{\text{DEF}}{=} (\alpha ? \top : \perp) \quad (15)$$

$$\partial(\varphi \wedge \psi) \stackrel{\text{DEF}}{=} \partial(\varphi) \wedge \partial(\psi) \quad (16)$$

$$\partial(\varphi \vee \psi) \stackrel{\text{DEF}}{=} \partial(\varphi) \vee \partial(\psi) \quad (17)$$

$$\partial(\neg\varphi) \stackrel{\text{DEF}}{=} \neg\partial(\varphi) \quad (18)$$

$$\partial(\mathbf{X}\psi) \stackrel{\text{DEF}}{=} \psi \quad (19)$$

$$\partial(\varphi \mathbf{U} \psi) \stackrel{\text{DEF}}{=} \partial(\psi) \vee (\partial(\varphi) \wedge (\varphi \mathbf{U} \psi)) \quad (20)$$

$$\partial(\varphi \mathbf{R} \psi) \stackrel{\text{DEF}}{=} \partial(\psi) \wedge (\partial(\varphi) \vee (\varphi \mathbf{R} \psi)) \quad (21)$$

Symbolic derivatives, as given above, lift the construction in [Vardi 1995] (see also [Kupferman 2018, Theorem 24]) as to be modulo \mathcal{A} . We use the following simplified rules for \mathbf{G} and \mathbf{F} (where $\perp \vee \phi \cong \phi$ and $\top \wedge \phi \cong \phi$ are applied implicitly): $\partial(\mathbf{G}\psi) \stackrel{\text{DEF}}{=} \partial(\psi) \wedge \mathbf{G}\psi$ and $\partial(\mathbf{F}\psi) \stackrel{\text{DEF}}{=} \partial(\psi) \vee \mathbf{F}\psi$. Correctness of the derivation rules will follow as a special case of Theorem 4 in Section 7.

We now link derivatives formally with ABWs. We write LTL^+ for the *positive* formulas in LTL where \neg does not occur. This is a standard normal form assumption and every formula in LTL can be translated into LTL^+ of the same size if members of \mathcal{A} are treated as units. The particular aspect with modulo \mathcal{A} is that *complement is propagated into \mathcal{A}* , i.e., for $\alpha \in \mathcal{A}$, the LTL formula $\neg\alpha$ becomes the predicate α^c in \mathcal{A} .

Definition 6.3 ($\text{LTL}^+(\mathcal{A})$). The *positive fragment* of $\text{LTL}\langle\mathcal{A}\rangle$ is obtained by eliminating any explicit use of \neg through de Morgan's laws and the standard equivalence preserving rules where $\alpha \in \mathcal{A}$ and $\varphi, \psi \in \text{LTL}$: $\neg\alpha \equiv \alpha^c$, $\neg\mathbf{X}\varphi \equiv \mathbf{X}\neg\varphi$, $\neg(\varphi \mathbf{U} \psi) \equiv \neg\varphi \mathbf{R} \neg\psi$, and $\neg(\varphi \mathbf{R} \psi) \equiv \neg\varphi \mathbf{U} \neg\psi$.

Definition 6.4 (M_ϕ for $\phi \in \text{LTL}^+(\mathcal{A})$). For $\phi \in \text{LTL}^+(\mathcal{A})$ let Q_ϕ denote the set containing \perp and \top and all *non-Boolean* subformulas of ϕ , i.e., predicates in \mathcal{A} , and all *modal* formulas. Let ϱ_ϕ denote $\lambda q. \partial(q)$ for $q \in Q_\phi$. Let F_ϕ contain all the \mathbf{R} -formulas and \top in Q_ϕ . Then $M_\phi \stackrel{\text{DEF}}{=} (\mathcal{A}, Q_\phi, \phi, \varrho_\phi, F_\phi)$. Observe that $\varrho_\phi : Q_\phi \rightarrow \mathbf{TTerm}\langle\mathcal{A}, \mathbb{B}^+(Q_\phi)\rangle$.

The key result from [Vardi 1995] ([Kupferman 2018, Theorem 24]) is lifted as to be modulo \mathcal{A} .

THEOREM 2 (VARDI DERIVATIVES MODULO THEORIES). $\forall \phi \in \text{LTL}^+(\mathcal{A}) : \mathcal{L}(M_\phi) = \mathcal{L}(\phi)$

Thus, the following invariant is preserved by *all* states ψ of M_ϕ because $\mathcal{L}_{M_\phi}(\psi) = \mathcal{L}(\psi)$.

COROLLARY 2 (LTL INVARIANCE). $\forall \phi \in \text{LTL}^+(\mathcal{A}) : \forall \psi \in Q_{M_\phi} : \mathcal{L}_{M_\phi}(\psi) = \mathcal{L}(\psi)$

Some states may become unreachable from ϕ through rewrites of ϱ_ϕ , that the definition of M_ϕ does not directly reflect, e.g., α and α^c are not relevant as states in Example 6.5.

Example 6.5. Let $\phi = \mathbf{G}(\mathbf{F}\alpha \wedge \mathbf{F}\alpha^c)$, where $\alpha \in \mathcal{A}$ is such that $\alpha \not\equiv \perp$ and $\alpha \not\equiv \top$. Then the transition function $\varrho = \varrho_\phi$ has the following reachable states and terms:

$$\varrho(\mathbf{F}\alpha) = (\alpha ? \top : \mathbf{F}\alpha), \quad \varrho(\mathbf{F}\alpha^c) = (\alpha^c ? \top : \mathbf{F}\alpha^c), \quad \varrho(\phi) = (\alpha ? \mathbf{F}\alpha^c \wedge \phi : \mathbf{F}\alpha \wedge \phi), \quad \varrho(\top) = \top$$

where $\partial(\mathbf{F}\alpha) \cong (\partial(\alpha) \vee \mathbf{F}\alpha) = (\alpha ? \top \vee \mathbf{F}\alpha : \perp \vee \mathbf{F}\alpha) \cong (\alpha ? \top : \mathbf{F}\alpha)$ (similarly for $\partial(\mathbf{F}\alpha^c)$), and

$$\begin{aligned} \partial(\phi) &\cong (\partial(\mathbf{F}\alpha \wedge \mathbf{F}\alpha^c) \wedge \phi) = (\partial(\mathbf{F}\alpha) \wedge \partial(\mathbf{F}\alpha^c) \wedge \phi) \\ &\cong ((\alpha ? \top : \mathbf{F}\alpha) \wedge (\alpha^c ? \top : \mathbf{F}\alpha^c) \wedge \phi) \\ &\cong (\alpha ? (\alpha^c ? \phi : \mathbf{F}\alpha^c \wedge \phi) : (\alpha^c ? \mathbf{F}\alpha \wedge \phi : \mathbf{F}\alpha \wedge \mathbf{F}\alpha^c \wedge \phi)) \\ &\cong (\alpha ? \mathbf{F}\alpha^c \wedge \phi : \mathbf{F}\alpha \wedge \phi) \end{aligned}$$

The resulting M_ϕ is shown in Figure 1 with $q_0 = \phi$, $q_1 = \mathbf{F}\alpha$ and $q_2 = \mathbf{F}\alpha^c$. \square

The following example illustrates the impact of cleaning modulo different element theories \mathcal{A} and how this is reflected in the computation of M_ϕ .

Example 6.6. Consider \mathcal{A} with an SMT solver, let x be of type *integer*, and let $\phi = (x < 1) \mathbf{R} (0 < x)$, stating that the condition that $x > 0$ is released when $x < 1$ becomes true. Then

$$\begin{aligned} \partial(\phi) &\cong \partial(0 < x) \wedge (\partial(x < 1) \vee \phi) = (0 < x ? \top : \perp) \wedge ((x < 1 ? \top : \perp) \vee \phi) \\ &\cong (0 < x ? \top : \perp) \wedge (x < 1 ? \top : \phi) \cong (0 < x ? (x < 1 ? \top : \phi) : \perp) \cong (0 < x ? \phi : \perp) \end{aligned}$$

where $\text{UNSAT}((0 < x) \sqcap (x < 1))$ is used for cleaning. I.e., in this case $(x < 1) \mathbf{R} (0 < x) \equiv \mathbf{G}(0 < x)$. \square

6.3 Alternation Elimination for LTL Modulo Theories

Combined together, Theorem 1 and Corollary 2 show the decidability of $\text{LTL}(\mathcal{A})$ for decidable \mathcal{A} . They allow us to directly use the \mathcal{A} algorithm on the transition terms resulting from the symbolic derivatives of $\text{LTL}(\mathcal{A})$. This will produce a nondeterministic Büchi automata from an LTL formula in a lazy manner, while safely applying many LTL-based rewrites.

THEOREM 3 (LTL INVARIANCE OF \mathcal{A}). $\forall \phi \in \text{LTL}^+(\mathcal{A}) : \forall q \in Q_{\mathcal{A}(M_\phi)} : \mathcal{L}_{\mathcal{A}(M_\phi)}(q) = \mathcal{L}(\wedge q)$

PROOF. Let $M = M_\phi$ and $N = \mathcal{A}(M)$. Fix $q = \langle U, V \rangle \in Q_N$. We show that $\mathcal{L}_N(q) = \mathcal{L}(\wedge q)$. We have that $\mathcal{L}_N(q) = \mathcal{L}_M(\wedge q) = \bigcap_{\psi \in U \cup V} \mathcal{L}_M(\psi) = \bigcap_{\psi \in U \cup V} \mathcal{L}(\psi) = \mathcal{L}(\wedge q)$ where the first equality holds by Theorem 1 and the third equality holds by Corollary 2 because $U \cup V \subseteq Q_M$. \square

States $\langle U, V \rangle$ in $Q_{\mathcal{A}(M_\phi)}$ can be reduced by Lemma 3. E.g., for any ϕ , if $\phi, \mathbf{G}(\phi \wedge _) \in V$ then ϕ can be eliminated from V because $\partial(\phi \wedge \mathbf{G}(\phi \wedge _)) \cong \partial(\mathbf{G}(\phi \wedge _))$, which means that the derivative (transition term) does not even need to be computed in this case, in order to know that Lemma 3 can be applied. In fact, this exact situation appeared twice implicitly in Example 5.1, where $M = M_{\mathbf{G}(\mathbf{F}\alpha \wedge \mathbf{F}\alpha^c)}$. Many similar rules can be used, where the one illustrated here is an instance of the subsumption rule in [Somenzi and Bloem 2000, ≤]. Such rules would clearly be out of reach if the LTL invariance property had been lost in translation. A full analysis of which rules in [Somenzi and Bloem 2000] preserve LTL invariance of \mathcal{A} is beyond the scope of this work.

7 LTL Extended with Regexes Modulo Theories

We now show the power of symbol derivatives by using them to combine the following two languages into one:

- ✓ **ERE** $\langle \mathcal{A} \rangle$, with derivatives summarized in Section 7.1;
- ✓ **LTL** $\langle \mathcal{A} \rangle$, as given in the previous section.

We first work with *finite* words in Σ^* , and later lift the semantics to *infinite* words in Σ^ω when we extend **LTL** $\langle \mathcal{A} \rangle$ with **ERE** $\langle \mathcal{A} \rangle$ in the combined language **RLTL** $\langle \mathcal{A} \rangle$ in Section 7.3.

Our focus here is on the *classical subset* of the extended regular expression operators supported in SPOT [Duret-Lutz 2024] and PSL [Eisner and Fisman 2006]. The language **RLTL** $\langle \mathcal{A} \rangle$ also has *regex complement* and *ω -closure*, which are not in SPOT/PSL but can be supported naturally with transition terms. In Section 7.7 we lift ω -regular languages to be modulo \mathcal{A} and show that **RLTL** $^+ \langle \mathcal{A} \rangle$ captures them precisely.

7.1 Derivatives of ERE as Transition Regexes

Recall the definition of **ERE** $\langle \mathcal{A} \rangle$ from Section 2.5 and let $R \in \mathbf{ERE} \langle \mathcal{A} \rangle$. The property *nullable*(R) means that $\epsilon \models R$ and is maintained as flag of each regex. Recall that $v \models R \Leftrightarrow v \in \mathcal{L}(R)$.

$$\begin{aligned} \text{nullable}(\alpha) &\stackrel{\text{DEF}}{=} \text{false} & \text{nullable}(\epsilon) &\stackrel{\text{DEF}}{=} \text{true} & \text{nullable}(R^*) &\stackrel{\text{DEF}}{=} \text{true} & \text{nullable}(\sim R) &\stackrel{\text{DEF}}{=} \neg \text{nullable}(R) \\ \text{nullable}(R_1 \uplus R_2) &\stackrel{\text{DEF}}{=} \text{nullable}(R_1) \vee \text{nullable}(R_2) & \text{nullable}(R_1 \cap R_2) &\stackrel{\text{DEF}}{=} \text{nullable}(R_1) \wedge \text{nullable}(R_2) \end{aligned}$$

The *derivative* of $R \in \mathbf{ERE}$ is denoted by $\delta(R) \in \mathbf{TTerm} \langle \mathcal{A}, \mathbf{ERE} \rangle$ [Stanford et al. 2021]. Recall that (1) and (2) automatically lift all the regex operators to **TTerm** $\langle \mathcal{A}, \mathbf{ERE} \rangle$, which means that the algebra of lift rules in [Stanford et al. 2021, Section 4.1] (dealing with incremental propagation of the operators in transition regexes) is irrelevant here.

$$\begin{aligned} \delta(\epsilon) &\stackrel{\text{DEF}}{=} \perp & \delta(R_1 \uplus R_2) &\stackrel{\text{DEF}}{=} \delta(R_1) \uplus \delta(R_2) \\ \delta(\alpha) &\stackrel{\text{DEF}}{=} (\alpha ? \epsilon : \perp) & \delta(R_1 \cap R_2) &\stackrel{\text{DEF}}{=} \delta(R_1) \cap \delta(R_2) \\ \delta(\sim R) &\stackrel{\text{DEF}}{=} \sim \delta(R) & \delta(R_1 \cdot R_2) &\stackrel{\text{DEF}}{=} \begin{cases} \delta(R_1) \cdot R_2 \uplus \delta(R_2), & \text{if } \text{nullable}(R_1); \\ \delta(R_1) \cdot R_2, & \text{otherwise.} \end{cases} \\ \delta(R^*) &\stackrel{\text{DEF}}{=} \delta(R) \cdot R^* & & \end{aligned} \tag{22}$$

In particular, the definition of $\delta(\alpha) = (\alpha ? \epsilon : \perp)$ differs crucially from $\partial(\alpha) = (\alpha ? \top : \perp)$ in **LTL** $\langle \mathcal{A} \rangle$. Moreover, $\neg(\alpha ? \top : \perp) \cong (\alpha ? \perp : \top)$ while $\sim(\alpha ? \top : \perp) \cong (\alpha ? \epsilon \uplus \top \top + : \top^*)$. In other words, while structurally similar, the derivatives (and their semantics) are fundamentally different.

The union operation \uplus over **ERE** is treated as an ACI operation in [Stanford et al. 2021], as the critical part of \cong over **ERE**, which implies that M_R , as defined next, is well-defined as a symbolic DFA modulo \mathcal{A} ($\text{DFA}_{\mathcal{A}}$) by having a *finite* state space.

Definition 7.1 (M_R for $R \in \mathbf{ERE} \langle \mathcal{A} \rangle$). $M_R = (\mathcal{A}, Q, R, \varrho, F)$ where Q is the least set of states containing R as the initial state and $\varrho : Q \rightarrow \mathbf{TTerm} \langle \mathcal{A}, Q \rangle$ is the transition function such that $\varrho(q) \cong \delta(q)$ and $\text{lvs}(\varrho(q)) \subseteq Q$. $F = \{q \in Q \mid \text{nullable}(q)\}$.

Let $M = M_R$. For all $a \in \Sigma, u \in \Sigma^*$ and $q \in Q$ let $\hat{\delta}(\epsilon, q) \stackrel{\text{DEF}}{=} q$ and $\hat{\delta}(a::u, q) \stackrel{\text{DEF}}{=} \hat{\delta}(u, \delta(q)[a])$. For all $q \in Q$ let $\mathcal{L}_M(q) \stackrel{\text{DEF}}{=} \{u \in \Sigma^* \mid \hat{\delta}(u, q) \in F\}$. Lemma 4 is implied by [Stanford et al. 2021, Theorem 4.3] and [Brzozowski 1964, Theorem 3.1].

LEMMA 4 ([STANFORD ET AL. 2021]). $\forall q \in Q : \mathcal{L}_M(q) = \mathcal{L}(q) \wedge \mathcal{L}(\varrho(q)[a]) = \text{D}_a(\mathcal{L}(q))$.

A state q is *alive* if $\mathcal{L}_M(q) \neq \emptyset$ else *dead*. Observe that a state $\hat{\delta}(u, R)$ is alive $\Leftrightarrow \exists x : ux \models R$.

7.2 One Step Lookahead

For $R \in \text{ERE}$ let $\text{OneStep}(R)$ be a predicate in \mathcal{A} that denotes $\mathcal{L}(R) \cap \Sigma$. The predicate $\text{OneStep}(R)$ plays a key role in the definition of derivatives for $\text{RLTL}(\mathcal{A})$ below, as a *one-step lookahead* such that, for all $a \in \Sigma$, $a \models \text{OneStep}(R) \Leftrightarrow a \models R$.

$$\begin{aligned} \text{OneStep}(R) &\stackrel{\text{DEF}}{=} \text{OneStep}'(\delta(R)) \\ \text{OneStep}'(R) &\stackrel{\text{DEF}}{=} \text{if } \text{nullable}(R) \text{ then } \top \text{ else } \perp \\ \text{OneStep}'((\alpha ? f : g)) &\stackrel{\text{DEF}}{=} (\alpha \sqcap \text{OneStep}'(f)) \sqcup (\alpha^c \sqcap \text{OneStep}'(g)) \end{aligned}$$

In particular, $\text{OneStep}(R) \equiv \perp \Leftrightarrow \mathcal{L}(R) \cap \Sigma = \emptyset$. The predicate $\text{OneStep}(R)$ is often used when $\delta(R)$ is used. Therefore, the predicate $\text{OneStep}'(f)$ is ideally cached (and simplified to \perp when unsat) with each transition regex f when f is constructed.

Example 7.2. We construct the transition terms of $M_{(\alpha\beta)^+}$, where $\alpha, \beta \in \mathcal{A}$, for all $r \in Q_{M_{(\alpha\beta)^+}}$.

r	$\delta(r)$	$\text{OneStep}(r)$	$\text{nullable}(r)$	$\text{DFA}_{\mathcal{A}} M_{(\alpha\beta)^+}$
$(\alpha\beta)^+$	$(\alpha ? \beta(\alpha\beta)^*)$	\perp	false	
$\beta(\alpha\beta)^*$	$(\beta ? (\alpha\beta)^*)$	β	false	
$(\alpha\beta)^*$	$(\alpha ? \beta(\alpha\beta)^*)$	\perp	true	

$\text{OneStep}(r)$ is shown for each state above. (We typically omit the state \perp to avoid clutter.) \(\square\)

7.3 RLTL

$\text{RLTL}(\mathcal{A})$ extends $\text{LTL}(\mathcal{A})$ with $\text{ERE}(\mathcal{A})$. Let $\phi \in \text{RLTL}(\mathcal{A})$, $R \in \text{ERE}(\mathcal{A})$, and $\alpha \in \mathcal{A}$.

$$\phi ::= \alpha \mid \neg\phi \mid \phi \vee \phi' \mid \phi \wedge \phi' \mid \mathbf{X}\phi \mid \phi \mathbf{U} \phi' \mid \phi \mathbf{R} \phi' \mid R \Diamond \rightarrow \phi \mid R \Box \rightarrow \phi \mid \{R\} \mid !\{R\} \mid R^\omega$$

The operator $\Diamond \rightarrow$ is *existential suffix implication*, $\Box \rightarrow$ is *(universal) suffix implication*, $\{R\}$ is *weak closure*, $!\{R\}$ is *negated weak closure*, and R^ω is ω -closure. The semantics of $\phi \in \text{RLTL}$ is a conservative extension of LTL and consistent with the semantics in SPOT [Duret-Lutz 2024], except that if R is *nullable* then in RLTL the formula $\{R\}$ (resp. $!\{R\}$) is equivalent to \top (resp. \perp) and $R^\omega, \sim R$ are not supported in PSL/SPOT. The SPOT formula $\{R\} (!\{R\})$ maps to $\{R \sqcap \top^+\} (!\{R \sqcap \top^+\})$ in RLTL . Definitions (9–14) carry over to RLTL and we add the new definitions:

$$w \models R \Diamond \rightarrow \phi \stackrel{\text{DEF}}{=} \exists i : \text{take}(i+1, w) \models R \wedge \text{drop}(i, w) \models \phi \quad (23)$$

$$w \models R \Box \rightarrow \phi \stackrel{\text{DEF}}{=} \forall i : \text{take}(i+1, w) \models R \Rightarrow \text{drop}(i, w) \models \phi \quad (24)$$

$$w \models \{R\} \stackrel{\text{DEF}}{=} \exists i : \text{take}(i, w) \models R \vee \forall i > 0 : \exists x : \text{take}(i, w) \cdot x \models R \quad (25)$$

$$w \models !\{R\} \stackrel{\text{DEF}}{=} w \not\models \{R\} \quad (26)$$

$$w \models R^\omega \stackrel{\text{DEF}}{=} w \in \mathcal{L}(R)^\omega \quad (27)$$

We use the relations $v <_\epsilon w \stackrel{\text{DEF}}{=} \exists i : v = \text{take}(i, w)$ and $v < w \stackrel{\text{DEF}}{=} \exists i > 0 : v = \text{take}(i, w)$ below.

Example 7.3. Let $R = \alpha^*\beta$ where $\llbracket \alpha \rrbracket = \{a\}$ and $\llbracket \beta \rrbracket = \{b\}$. Then $\nexists u <_\epsilon a^\omega : u \in \mathcal{L}(R)$, however $\forall u < a^\omega : \exists x : ux \in \mathcal{L}(R)$ (namely for $x = b$) – i.e., for all $u < a^\omega$, $\hat{\delta}(u, R)$ is alive. So $a^\omega \models \{R\}$. \(\square\)

It follows from the definitions that $R \Box \rightarrow \phi \equiv \neg(R \Diamond \rightarrow \neg\phi)$ and trivially that $!\{R\} \equiv \neg\{R\}$, i.e., (24) and (26) are duals of (23) and (25).

Definition 7.4 (RLTL⁺). The *positive fragment of RLTL* is obtained by using Definition 6.3, and for $R \in \text{ERE}$ and $\phi, \psi \in \text{RLTL}$: $\neg(R \Diamond \rightarrow \phi) \equiv R \Box \rightarrow \neg\phi$, $\neg(R \Box \rightarrow \phi) \equiv R \Diamond \rightarrow \neg\phi$, $\neg\{R\} \equiv !\{R\}$, and $\neg!\{R\} \equiv \{R\}$. **RLTL⁺ disallows $\neg(R^\omega)$.**

Example 7.5. Recall $\sim R \Box \rightarrow \perp$ from Section 1.3 that belongs to RLTL^+ . So

$$\begin{aligned} w \models \sim R \Box \rightarrow \perp &\Leftrightarrow \forall i : \text{take}(i+1, w) \not\models \sim R \vee \text{drop}(i, w) \models \perp \Leftrightarrow \forall i > 0 : \text{take}(i, w) \not\models \sim R \\ &\Leftrightarrow \forall i > 0 : \text{take}(i, w) \models R \end{aligned}$$

and thus $w \models \sim R \Box \rightarrow \perp \Leftrightarrow \forall v < w : v \models R$. E.g., if a regex r is *bounded* (only accepts words of *bounded length*, any regex that is both $*$ -free and \sim -free is bounded) and $\lfloor r \rfloor$ is the *prefix closure regex* of r , then $\sim(r* \cdot \lfloor r \rfloor) \Box \rightarrow \perp \equiv r^\omega$. Since $\neg(r^\omega)$ is not supported in $\text{RLTL}^+(\mathcal{A})$, $\sim(r* \cdot \lfloor r \rfloor) \Diamond \rightarrow \top$ can be used instead of $\neg(r^\omega)$ if r is bounded. \square

For $M \in \text{ABW}_{\mathcal{A}}$ and $\phi \in \text{RLTL}(\mathcal{A})$ let $M \models \phi \stackrel{\text{DEF}}{=} \mathcal{L}(M) \subseteq \mathcal{L}(\phi)$.

7.4 Trace Analysis of Cloud Services

Our primary application of $\text{RLTL}(\mathcal{A})$ is to analyze traces of *cloud services*. The elements in Σ are called *actions* in this context, that are pairs (req, res) of *JSON values* where *req* represents an *http request* and *res* represents an *http response*. SMT solvers like Z3 [de Moura and Bjørner 2008] can natively reason about JSON values as they can be represented by nested data types.

Note that a JSON value here is a structured representation of an *http request* or *response* that is itself just flat text. The basic fields of a JSON value are usually *strings*, that are sequences of characters of type *Unicode* in Z3. JSON fields are often associated with additional type constraints in form of regular expressions, e.g., that a *key* field must be a nonempty string of digits, so it must match the regex d^+ . Thus, *regexes modulo Unicode* are part of \mathcal{A} , while *regexes modulo \mathcal{A}* are used to specify action patterns in $\text{RLTL}(\mathcal{A})$, as illustrated below.

Since cloud services run continuously their trace semantics is naturally described in terms of *action streams*. (In reality, there can also be several requests in-flight simultaneously, but here we let each request *req* be paired with an immediate response *res*.)

A model of the cloud service is given or is being learned as an $\text{NBW}_{\mathcal{A}}$ M . If we have some RLTL property ψ that is expected to hold, i.e., $M \models \psi$ is expected, but $\mathcal{L}(M \times \mathcal{A}(M_{\neg\psi})) \neq \emptyset$ we can extract a counterexample from $M \times \mathcal{A}(M_{\neg\psi})$. If the counterexample turns out to be spurious, by evaluating it against the actual service, it can be used to refine M (during learning).

For example, in some services requests and responses must have equal IDs in the same action, in other words $M \models \mathbf{G}(\text{ID}(\text{req}) = \text{ID}(\text{res}))$ must hold. Let FAIL denote failed actions and let $\text{SUCC} = \text{FAIL}^c$. A failure typically means that *res* contains an *http status code* in the range ≥ 400 , but it may also be a request specific error message, e.g., that a null reference exception was thrown, or that the *etag* values in *req* and *res* do not match when expected to match for certain operations.

We consider a subset of the *Azure App Configuration* service [Microsoft 2023] for basic key operations. For a *fixed* key, we use the following predicates on *req*: D = the operation is *DeleteKey*; L = the operation is *LockKey*; U = the operation is *UnlockKey*. Then the following formula, say ψ ,

$$\top * \cdot (\text{L} \sqcap \text{SUCC}) \cdot (\text{U}^c \sqcup \text{FAIL}) * \cdot \text{D} \Box \rightarrow \text{FAIL}$$

states that a locked key cannot be deleted. It describes the following action pattern: if at some point the key is successfully locked ($\text{L} \sqcap \text{SUCC}$) and there are no subsequent successful unlock actions ($\text{U}^c \sqcup \text{FAIL}$)* before a delete request D then D must fail. Observe that D and FAIL in ψ apply to the same action w_i in any action stream w – the last action in $\text{take}(i+1, w)$ is also the first action in $\text{drop}(i, w)$ in the semantics of $\Box \rightarrow$ (24). $M_{\neg\psi}$ is illustrated in Figure 2.

Patterns such as the one above are often most naturally expressed by using regexes and suffix implications, that also illustrates a core advantage over the $\text{LTL}(\mathcal{A})$ fragment. It is often unclear and difficult to decide if a property can or cannot be expressed in $\text{LTL}(\mathcal{A})$. Even when a property can be expressed in $\text{LTL}(\mathcal{A})$ it can be difficult to understand the resulting formula. As it happens,

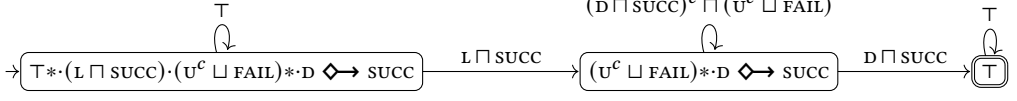


Fig. 2. $M_{\neg\psi}$ where $\psi = T * (L \sqcap \text{SUCC}) \cdot (U^c \sqcup \text{FAIL}) * D \blacklozenge \rightarrow \text{SUCC}$ as in Section 7.4.

the formula ψ above can be written as the following equivalent formula in $\text{LTL}\langle\mathcal{A}\rangle$, say ϕ ,

$$\mathbf{G}((L \wedge \text{SUCC}) \rightarrow \mathbf{X}((U \wedge \text{SUCC}) \mathbf{R} (D \rightarrow \text{FAIL})))$$

stating that, whenever the key has been successfully locked then, consequently, successful unlock releases the condition that delete fails. The $\text{LTL}\langle\mathcal{A}\rangle$ symbolic derivative of $\neg\phi$ is as follows:

$$\begin{aligned} \partial(\neg\phi) &\cong (L \sqcap \text{SUCC} ? ((U \rightarrow \text{FAIL}) \mathbf{U} (D \wedge \text{SUCC})) \vee \neg\phi : \neg\phi) \\ \partial((U \rightarrow \text{FAIL}) \mathbf{U} (D \wedge \text{SUCC})) &\cong (D \sqcap \text{SUCC} ? \top : (U^c \sqcup \text{FAIL} ? (U \rightarrow \text{FAIL}) \mathbf{U} (D \wedge \text{SUCC}))) \end{aligned}$$

The automaton $M_{\neg\phi}$ is illustrated in Figure 3 and is clearly equivalent to $M_{\neg\psi}$ in Figure 2.

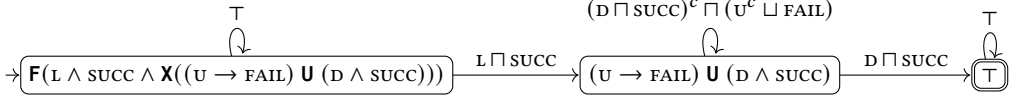


Fig. 3. $M_{\neg\phi}$ where $\phi = \mathbf{G}((L \wedge \text{SUCC}) \rightarrow \mathbf{X}((U \wedge \text{SUCC}) \mathbf{R} (D \rightarrow \text{FAIL})))$ as in Section 7.4.

Observe that the predicates here are not just propositions but predicates over the underlying JSON values. For example, a request cannot simultaneously be a delete operation and an unlock operation, i.e., $\text{UNSAT}(D \wedge U)$. Therefore, any bitblasting to propositions, say p_D and p_U in SPOT, would require the theory constraint $\neg(p_D \wedge p_U)$, among all the other required theory constraints.

7.5 Derivatives of RLTL

Derivatives for RLTL are defined as terms in $\text{TTerm}\langle\mathcal{A}, \text{RLTL}\rangle$ reusing (15–21) and the rules:

$$\partial(R \blacklozenge \phi) \stackrel{\text{DEF}}{=} (\text{OneStep}(R) ? \partial(\phi) : \perp) \vee (\delta(R) \blacklozenge \phi) \quad (28)$$

$$\partial(R \blacklozenge \phi) \stackrel{\text{DEF}}{=} (\text{OneStep}(R) ? \partial(\phi) : \top) \wedge (\delta(R) \blacklozenge \phi) \quad (29)$$

$$\partial(\{R\}) \stackrel{\text{DEF}}{=} \text{if nullable}(R) \text{ then } \top \text{ else } \{\delta(R)\} \quad (30)$$

$$\partial(\{!R\}) \stackrel{\text{DEF}}{=} \text{if nullable}(R) \text{ then } \perp \text{ else } \{!\delta(R)\} \quad (31)$$

$$\partial(R^\omega) \stackrel{\text{DEF}}{=} \partial(R \blacklozenge \mathbf{X}R^\omega) \quad (32)$$

where $r \blacklozenge \phi$, $r \blacklozenge \phi$, $\{r\}$, and $\{!r\}$ are lifted to $\text{TTerm}\langle\mathcal{A}, \text{RLTL}\rangle$ by (2). Rules (28–31) seamlessly combine the derivatives of LTL (15–21) with derivatives of ERE (22). Correctness is proved in Theorem 4. Several derived laws can be inlined as immediate rewrites into the definition of ∂ , such as $(\perp \blacklozenge \phi) \cong (\epsilon \blacklozenge \phi) \cong \{\perp\} \cong \perp$, and if R is nullable then $\{R\} \cong \top$. Analogous rewrites apply to their duals \blacklozenge and $\{!R\}$. Note that if $\text{OneStep}(R) \equiv \perp$ then $(\text{OneStep}(R) ? \partial(\phi) : \perp) \cong \perp$ and $(\text{OneStep}(R) ? \partial(\phi) : \top) \cong \top$ whereby the definitions (28) and (29) simplify accordingly.

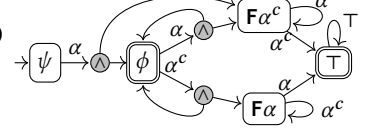
Let $\text{Acc}(\text{RLTL}^+)$ denote the subset of all the following formulas of RLTL^+ : \top , all \mathbf{R} formulas, all \blacklozenge formulas, all $\{R\}$ formulas where R is *alive*, all $\{!R\}$ formulas where R is *dead*, and all R^ω formulas.

Definition 7.6 (M_ϕ for $\phi \in \text{RLTL}^+(\mathcal{A})$). $M_\phi = (\mathcal{A}, Q, \phi, \varrho, Q \cap \text{Acc}(\text{RLTL}^+))$ where $Q \subseteq \text{RLTL}^+$ is the least set such that $\phi \in Q$ and if $q \in Q$ then $\varrho(q) \cong \partial(q)$ and if p is a non-Boolean subformula of a leaf of $\varrho(q)$ then $p \in Q$.

M_ϕ is well-defined (Q above is finite) because all the M_R are finite by Lemma 4. In contrast to LTL derivatives, where all the original subformulas suffice, for $\phi \in \mathbf{RLTL}$, M_ϕ also gets new formulas in suffix implications and closures, e.g., in every leaf $r \diamond \psi$ of $\delta(R) \diamond \psi$, r is some leaf of $\delta(R)$ and $r \diamond \psi$ is typically not a subformula of ϕ . (See Figure 2.)

Example 7.7. Consider the RLTL formula $\psi = \alpha \diamond \phi$ with $\phi = \mathbf{G}(\mathbf{F}\alpha \wedge \mathbf{F}\alpha^c)$ and recall Example 6.5. Here $\delta(\alpha) = (\alpha ? \varepsilon)$. Since $\text{OneStep}(\alpha) = \alpha$, by using (28), we get that

$$\begin{aligned} \partial(\psi) &= (\alpha ? \partial(\phi)) \vee (\delta(\alpha) \diamond \phi) \cong (\alpha ? \partial(\phi)) \vee (\alpha ? (\varepsilon \diamond \phi)) \\ &\cong (\alpha ? \partial(\phi)) \vee (\alpha ? \perp) \cong (\alpha ? \partial(\phi)) \\ &\cong (\alpha ? (\alpha ? \mathbf{F}\alpha^c \wedge \phi : \mathbf{F}\alpha \wedge \phi)) \cong (\alpha ? \mathbf{F}\alpha^c \wedge \phi) \end{aligned}$$



The complete $ABW_{\mathcal{A}}$ is depicted on the right. $\mathcal{A}E$ can be used to obtain an $NBW_{\mathcal{A}}$ in this case. \square

Example 7.8. Let $\alpha, \beta, \gamma \in \mathcal{A}$ and consider $\phi = (\alpha\beta)^+ \diamond \mathbf{G}\gamma$. The transition terms that arise from ϕ give rise here to the ABW_{M_ϕ} as shown below, where we reused the regex derivatives from the earlier Example 7.2. Observe that M_ϕ also happens to be nondeterministic because conjunctions do not arise. Simplifications are directly inlined for \perp as the zero of \wedge and the unit of \vee .

q	$\partial(q)$	$NBW_{\mathcal{A}} M_\phi$
$q_0 = (\alpha\beta)^+ \diamond \mathbf{G}\gamma$	$(\perp ? \partial(\mathbf{G}\gamma)) \vee ((\alpha ? \beta(\alpha\beta)^*) \diamond \mathbf{G}\gamma)$ $\cong (\alpha ? \beta(\alpha\beta)^* \diamond \mathbf{G}\gamma)$	
$q_1 = \beta(\alpha\beta)^* \diamond \mathbf{G}\gamma$	$(\beta ? \partial(\mathbf{G}\gamma)) \vee (\beta ? (\alpha\beta)^* \diamond \mathbf{G}\gamma)$ $\cong (\beta \sqcap \gamma ? \mathbf{G}\gamma) \vee (\beta ? (\alpha\beta)^* \diamond \mathbf{G}\gamma)$	
$q_2 = (\alpha\beta)^* \diamond \mathbf{G}\gamma$	$(\alpha ? \beta(\alpha\beta)^* \diamond \mathbf{G}\gamma)$	
$q_3 = \mathbf{G}\gamma$	$(\gamma ? \mathbf{G}\gamma)$	

If $\beta \sqcap \gamma \neq \perp$ then M_ϕ is actually nondeterministic because the outgoing transition guards from q_1 overlap, i.e., the disjunction $q_2 \vee q_3$ is the leaf $\partial(q_1)[a]$ for any $a \models \beta \sqcap \gamma$. \square

Example 7.9. Let α and β be two predicates in \mathcal{A} such that $\llbracket \alpha \rrbracket = \{a\}$ and $\llbracket \beta \rrbracket = \{b\}$ where $a \neq b$. Let $R = (\alpha \cdot \top)^* \cdot \beta$. The following table shows the transitions that arise in the $NBW_{M_{\{R\}}}$, where $\text{UNSAT}(\alpha \sqcap \beta)$ is used to clean $\partial(\{R\})$. Observe that $\alpha^c \sqcap \beta \equiv \beta$. The table also shows $M_{! \{R\}}$ where $! \{(\beta ? \varepsilon)\} = (\beta ? ! \{\varepsilon\} : ! \{\perp\}) \cong (\beta ? \perp : \top) \cong (\beta^c ? \top)$.

q	$\partial(q)$	$NBW_{M_{\{R\}}}$	$NBW_{M_{! \{R\}}}$
$\{R\}$	$(\alpha ? \{\top \cdot R\} : (\beta ? \{\varepsilon\}))$		
$\{\top \cdot R\}$	$(\top ? \{R\}) \cong \{R\}$		
$\top (\cong \{\varepsilon\})$	\top		

Then $a^\omega \models \{R\}$ because $\forall u < a^\omega : \exists x : u \cdot x \models R$. Note also that $\mathcal{L}(\mathbf{G}(\beta^c) \wedge \{R\}) = \mathcal{L}((\alpha\beta^c)^\omega)$ where α holds in all even positions – a classical example not expressible in LTL [Wolper 1983]. \square

Example 7.10. With $\alpha, \beta \in \mathcal{A}$ let $\phi = (\alpha\beta)^\omega$. We get the following NBW :

q	$\partial(q) = \partial(r \diamond \mathbf{X}\phi)$	$\text{OneStep}(r)$	NBW_{M_ϕ}
ϕ	$\partial(\alpha\beta \diamond \mathbf{X}\phi) \cong (\delta(\alpha\beta) \diamond \mathbf{X}\phi)$ $\cong ((\alpha ? \beta) \diamond \mathbf{X}\phi) = (\alpha ? \beta \diamond \mathbf{X}\phi : \perp \diamond \mathbf{X}\phi)$ $\cong (\alpha ? \beta \diamond \mathbf{X}\phi)$	\perp	
$\beta \diamond \mathbf{X}\phi$	$(\beta ? \partial(\mathbf{X}\phi)) \vee (\delta(\beta) \diamond \mathbf{X}\phi)$ $= (\beta ? \phi) \vee (\beta ? \varepsilon \diamond \mathbf{X}\phi : \perp \diamond \mathbf{X}\phi)$ $\cong (\beta ? \phi) \vee (\beta ? \perp : \perp) \cong (\beta ? \phi) \vee \perp \cong (\beta ? \phi)$	β	

where many rewrite rules for \cong are being used. \square

Recall that $w_{..i} = \text{take}(i + 1, w)$ and $w_{i..} = \text{drop}(i, w)$. A distributivity law that can be very practical is that $(R_1 \uplus R_2) \blacktriangleright \phi \equiv (R_1 \blacktriangleright \phi) \vee (R_2 \blacktriangleright \phi)$, because for all $w \in \Sigma^\omega$ it holds that

$$\begin{aligned} \exists i : w_{..i} \models R_1 \uplus R_2 \wedge w_{i..} \models \phi &\Leftrightarrow \exists i : (w_{..i} \models R_1 \wedge w_{i..} \models \phi) \vee (w_{..i} \models R_2 \wedge w_{i..} \models \phi) \\ &\Leftrightarrow (\exists i : (w_{..i} \models R_1 \wedge w_{i..} \models \phi)) \vee (\exists i : (w_{..i} \models R_2 \wedge w_{i..} \models \phi)) \end{aligned}$$

Theorem 4 describes the semantics of RLTL in terms of languages and derivatives. It shows that the definition of derivatives correctly captures the intended semantics and is proved formally in Lean in Section 7.6 along with all required background theory. Theorem 4 does not yet imply any formal relationship with the corresponding ABW semantics, which is discussed in Section 7.7.

THEOREM 4 (DERIVATION). $\forall \phi \in \mathbf{RLTL}(\mathcal{A}), a \in \Sigma, w \in \Sigma^\omega : a::w \models \phi \Leftrightarrow w \models \partial(\phi)[a]$

The distributivity law of \blacktriangleright over union can be lifted and applied to derivatives through Theorem 4. This is related to adapting the construction in [Antimirov 1996] for RLTL that can avoid exponential blowup of the state space during incremental application of derivatives. Intuitively this amounts to unfolding a regex into an NFA instead of a DFA while distributing \blacktriangleright over union. A further derived law that follows is that

$$(R_1 \uplus R_2) \blacktriangleright \phi \equiv \neg((R_1 \uplus R_2) \blacktriangleright \neg \phi) \equiv \neg(R_1 \blacktriangleright \neg \phi) \wedge \neg(R_2 \blacktriangleright \neg \phi) \equiv (R_1 \blacktriangleright \phi) \wedge (R_2 \blacktriangleright \phi)$$

Such laws can also be formalized and proved correct in Lean.

7.6 Formalization in Lean

In this section we describe some of the key features of the Lean formalization, which focuses on the correctness of the derivative-based algorithm for RLTL given in Theorem 4.

Throughout the formalization we indicate the type of predicates by \mathcal{A} (A in Lean) and the type of alphabet as Σ . The two are connected by the notion of EBA $(\Sigma, \mathcal{A}, \models, \perp, \top, \sqcup, \sqcap, ^c)$ as formally presented in [Zhuchko et al. 2024] using type classes, where Σ is marked as an out-param and is thus automatically inferred from \mathcal{A} . The inductive type \mathbf{TTerm} represents $\mathbf{TTerm}(\mathcal{A}, \mathcal{B})$ where \mathcal{A} is the condition type and \mathcal{B} (B in Lean) the leaf type.

inductive $\mathbf{TTerm} \text{ (A B : Type) : Type}$ where

| Leaf : $B \rightarrow \mathbf{TTerm} \text{ A B}$

| Node (condition : A) (lthen : $\mathbf{TTerm} \text{ A B}$) (l_else : $\mathbf{TTerm} \text{ A B}$) : $\mathbf{TTerm} \text{ A B}$

The evaluation of a transition term f for an element a of type Σ is defined as follows in Lean and the notation $f[a]$ in Lean corresponds precisely to the math notation $f[a]$.

def evaluation (a : Σ) (f : $\mathbf{TTerm} \text{ A B}$) : B :=

match f with

| Leaf b => b

| Node p f g => if a \models p then evaluation a f else evaluation a g

notation f "[a "]" => evaluation a f

The type $\mathbf{Stream}' \Sigma$ represents Σ^ω as functions from \mathbb{N} to Σ , and is provided by the Lean standard library along with drop, take, head, tail, and get functions where (get w i) returns w_i .

We lift unary and binary operations to \mathbf{TTerm} as presented in Section 3. The two crucial lemmas used throughout the formalization show the correctness of the lifting operations:

theorem liftU (op : $B \rightarrow B'$) {f : $\mathbf{TTerm} \text{ A B}$ } (x : Σ) :

(lift_unary op f)[x] = op (f [x])

theorem liftB (op : $B \rightarrow B \rightarrow B'$) (f g : $\mathbf{TTerm} \text{ A B}$) (x : Σ) :

(lift_binary op f g)[x] = op (f [x]) (g [x])

For example, if $f, g \in \mathbf{Term}(\mathcal{A}, \mathcal{B})$ and o is a binary operation over \mathcal{B} then the math expression $(f \circ g)$ corresponds in Lean to the expression `(lift_binary o f g)`. The `liftU` and `liftB` lemmas are distribution laws for `op` and evaluation that correspond to (3) in Section 3.

7.6.1 Regular Languages. Before we can introduce semantics for infinite words, we have to define the semantics for finite words. We base the general structure of our approach on the formalization presented in [Zhuchko et al. 2024], which defines match semantics for regular expressions with lookarounds. In our context, we simplify this by excluding the cases for lookarounds in the type ERE. For a word v and a regular expression r , the match semantics $v \models r$ corresponds to the classical semantics $v \in \mathcal{L}(r)$. The theorems below establish the equivalence between derivatives and the classical language-based semantics, proved by induction on the match length:

theorem equivalenceNull $\{r : \text{ERE } A\} : [] \models r \leftrightarrow \text{nullable } r$

theorem equivalenceDer $\{r : \text{ERE } A\} : a :: v \models r \leftrightarrow v \models (\delta r)[a]$

The type of finite words Σ^* is in Lean represented by `List Σ` where ϵ is the empty list `[]` in Lean.

7.6.2 ω -Regular Languages. We now show how we formalize the notion of ω -regular language presented by a regular expression R . Since a language is a set of infinite words, we define a stream w to be in the ω -closure of R if it can be partitioned into subwords of non-zero lengths such that each subword is in the language of R . We formalize this idea by requiring the existence of a stream of natural numbers, called deltas, which indicates for any i the length of the i -th subword minus one. Each subword must then be accepted by R . A graphical description is given as follows:

$$\begin{array}{ccccccc} \text{deltas} = & 2, & & 4, & & 1, & \dots \\ & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & \\ w = & a, & b, & c, & d, & e, & f, & g, & h, & i, & l, & m, & \dots \end{array}$$

We start by defining a function `getWordStart` which, given $i \in \mathbb{N}$, returns the starting position of the i -th subword in the stream w by summing up the values of the stream of deltas up to i .

```
def getWordStart (deltas : Stream'N) (i : N) : N :=
  match i with
  | 0 => 0
  | .succ j => (getWordStart (tail deltas) j) + (head deltas) + 1
```

Note that in order to enforce the invariant that each subword has non-zero length, and since the stream `deltas` represents lengths minus 1, each individual delta is incremented by 1 – the calculation for $i > 0$ is precisely $\sum_{j < i} 1 + \Delta_j$ in Section 2.3 where Δ corresponds to `deltas`. This explains the `(head deltas) + 1` in the inductive case.

The central definition is `IsDeltasOmegaLanguage`, which expresses the idea that each subword in w indicated by a given delta is recognized by r .

```
def IsDeltasOmegaLanguage (w : Stream'Σ) (r : ERE A) (deltas : Stream'N) : Prop :=
  ∀ (i : N),
    let start := getWordStart deltas i
    let len := (get deltas i) + 1
    take len (drop start w) ⊨ r
```

that for $i > 0$ corresponds to $\text{take}(1 + \Delta_i, \text{drop}(\sum_{j < i} 1 + \Delta_j, w)) \in \mathcal{L}(r)$. Finally, `InOmegaLanguage` checks for the existence of a `deltas` that correctly partitions w into recognized subwords.

```
def InOmegaLanguage (w : Stream'Σ) (r : ERE A) : Prop :=
  ∃ (deltas : Stream'N), IsDeltasOmegaLanguage w r deltas
infixr:40 " ∈* " => InOmegaLanguage
```

In summary, we have that $w \in* r \Leftrightarrow w \in \mathcal{L}(r)^\omega$. We use the following theorem.

theorem regexOmegaClosure $\{r : \text{ERE } A\} : w \in* r \leftrightarrow \exists i > 0, \text{ take } i \ w \models r \wedge \text{drop } i \ w \in* r$

7.6.3 RLTL Semantics. In Lean, we represent the models relation \models as a binary predicate `models` between streams and RLTL formulas. The relation represents \models as defined by the rules (9–13) and (23–27), and uses the models relation \models of A as well as the models relation \models of ERE A.

```
def models (w : Stream'Σ) : RLTL A → Prop
| RLTL.Pred p => head w ⊢ p
| ¬I φ      => ¬ models w φ
| φ ∧I ψ    => models w φ ∧ models w ψ
| φ ∨I ψ    => models w φ ∨ models w ψ
| φ →I ψ    => models w φ → models w ψ
| X φ       => models (tail w) φ
| φ U ψ     => ∃ j, models (drop j w) ψ ∧ ∀ i < j, models (drop i w) φ
| φ R ψ     => ∃ j, models (drop j w) φ ∧ ∀ k ≤ j, models (drop k w) ψ
              ∨ ∀ i, models (drop i w) ψ
| r ◇→ φ    => ∃ i, take (i + 1) w ⊢ r ∧ models (drop i w) φ
| r □→ φ    => ∀ i, take (i + 1) w ⊢ r → models (drop i w) φ
| { r }     => (∃ i, take i w ⊢ r) ∨ (∀ i > 0, ∃ x, (take i w) ++ x ⊢ r)
| rω       => w ∈* r

notation:52 lhs:53 " |= " rhs:53 => models lhs rhs
```

Similarly to the case of match semantics for regular languages [Zhuchko et al. 2024], we define `models` as a function rather than an inductive predicate to avoid the negative occurrence of the relation for the $\neg_I \phi$ constructor. Note that in Lean, the negation operator $\neg p$ is defined as $p \rightarrow \perp$. In order to keep the definitions constructive, we introduce implication \rightarrow_I as a built-in operator in Lean. The negative weak closure operator is defined in Lean as just syntactic sugar for $\neg_I\{r\}$.

The definition of derivatives of RLTL formulas closely follows the definitions (15–21) and definitions (28–32) in Section 7.3. To illustrate this, we present the definition of the derivative for the existential suffix implication and weak closure operators:

```
def derivative (r : RLTL A) : TTerm A (RLTL A) :=
| RLTL.Pred p => Node p (Leaf (Pred ⊤)) (Leaf (Pred ⊥))
...
| r ◇→ φ    => let lhs := Node (OneStep r) (derivative φ) (Leaf (Pred ⊥))
               lift_binary (· ∨I ·) lhs (lift_unary (fun x => x ◇→ φ) (δ r))
| { r }     => if nullable r then Leaf (Pred ⊤) else (lift_unary ({ . }) (δ r))
prefix:max " ∂ " => RLTL.derivative
```

Another key definition is the `OneStep` predicate, which is essential for defining the derivative of an RLTL formula. This is defined in terms of a helper function `OneStep'` that computes the match semantics of a derivative of an ERE regex. `OneStep` is defined as `OneStep'` of the derivative of `r`.

```
def OneStep' (tr : TTerm A (ERE A)) : A :=
match tr with
| Leaf r      => if nullable r then ⊤ else ⊥
| Node p f g => (p ⊓ OneStep' f) ⊔ (pc ⊓ OneStep' g)
def OneStep (r : ERE A) := OneStep' (δ r)
```

The main correctness and completeness property of `OneStep`, as well as the invariants needed for `OneStep'`, can be identified as follows:

```
theorem denoteOneStep' {f : TTerm A (ERE A)} : [] ⊢ f[a] ↔ a ⊢ OneStep' f
theorem denoteOneStep {r : ERE A} : [a] ⊢ r ↔ a ⊢ OneStep r
```

When handling the dual temporal operators **U** and **R**, the following expansion laws prove to be particularly useful:

```
theorem expansion_release {φ ψ : RLTL A} : w |= φ R ψ ↔ w |= ψ ∧I (φ ∨I X(φ R ψ))
theorem expansion_until  {φ ψ : RLTL A} : w |= φ U ψ ↔ w |= ψ ∨I (φ ∧I X(φ U ψ))
```

The main correctness theorem for derivatives, given in Theorem 4 states that taking the derivative of an RLTL formula and then evaluating it on a single element of the domain preserves the match semantics.

theorem derivation $\{\phi : \text{RLTL } A\} : a::w \models \phi \leftrightarrow w \models (\partial \phi)[a]$

The proof follows by induction on ϕ , using the correctness and completeness of OneStep and of the standard match semantics of regular expressions on finite words. We highlight some key cases of the proof and use the math formulation when presenting these for conciseness.

The proof of the case of existential suffix implication can be outlined as follows, although in Lean the two directions of the equivalence are proved as separate implications and use also secondary induction over i . Recall that $w_{..i} = \text{take}(i+1, w)$ and $w_{i..} = \text{drop}(i, w)$.

$$\begin{aligned}
 a::w \models r \Diamond \psi &\Leftrightarrow \exists i : (a::w)_{..i} \models r \wedge (a::w)_{i..} \models \psi \\
 &\Leftrightarrow (a::w)_{..0} \models r \wedge a::w \models \psi \vee \exists i > 0 : (a::w)_{..i} \models r \wedge (a::w)_{i..} \models \psi \\
 &\stackrel{\dagger}{\Leftrightarrow} a \models \text{OneStep}(r) \wedge a::w \models \psi \vee \exists i : w_{..i} \models \delta(r)[a] \wedge w_{i..} \models \psi \\
 &\Leftrightarrow a \models \text{OneStep}(r) \wedge a::w \models \psi \vee w \models \delta(r)[a] \Diamond \psi \\
 &\stackrel{\text{IH}}{\Leftrightarrow} a \models \text{OneStep}(r) \wedge w \models \partial(\psi)[a] \vee w \models \delta(r)[a] \Diamond \psi \\
 &\Leftrightarrow w \models (\text{OneStep}(r) ? \top)[a] \wedge w \models \partial(\psi)[a] \vee w \models \delta(r)[a] \Diamond \psi \\
 &\Leftrightarrow w \models (\text{OneStep}(r) ? \partial(\psi))[a] \vee w \models \delta(r)[a] \Diamond \psi \\
 &\Leftrightarrow w \models (\text{OneStep}(r) ? \partial(\psi))[a] \vee \delta(r)[a] \Diamond \psi \\
 &\Leftrightarrow w \models ((\text{OneStep}(r) ? \partial(\psi)) \vee \delta(r) \Diamond \psi)[a] \\
 &\Leftrightarrow w \models \partial(r \Diamond \psi)[a]
 \end{aligned}$$

In step \dagger the proof uses the theorem equivalenceDer as well as the theorem denoteOneStep. Step IH indicates use of the induction hypothesis over RLTL formulas. The remaining proof steps make, on multiple occasions, use of liftU, liftB for lifting operations over transition terms that is implicit in the math formulation. It also uses the core properties of streams.

The proof of the case of weak closure $\{r\}$ is largely based on theorems equivalenceDer and equivalenceNull and does not use the induction hypothesis over RLTL formulas. Here we present it as further two separate subcases when r is nullable and when r is not nullable and use the more relaxed math notation to avoid liftU and liftB that are implicit.

If $\text{nullable}(r) = \text{true}$ then

$$a::w \models \{r\} \Leftrightarrow a::w \models \top \Leftrightarrow w \models \top[a] \Leftrightarrow w \models \partial(\{r\})[a].$$

If $\text{nullable}(r) = \text{false}$ then

$$\begin{aligned}
 a::w \models \{r\} &\Leftrightarrow \exists i : \text{take}(i, a::w) \models r \vee \forall i > 0 : \exists x : \text{take}(i, a::w) \cdot x \models r \\
 &\Leftrightarrow \exists i > 0 : \text{take}(i, a::w) \models r \vee \forall i > 0 : \exists x : \text{take}(i, a::w) \cdot x \models r \\
 &\Leftrightarrow \exists i : \text{take}(i, w) \models \delta(r)[a] \vee \forall i : \exists y : \text{take}(i, w) \cdot y \models \delta(r)[a] \\
 &\stackrel{\ddagger}{\Leftrightarrow} \exists i : \text{take}(i, w) \models \delta(r)[a] \vee \forall i > 0 : \exists x : \text{take}(i, w) \cdot x \models \delta(r)[a] \\
 &\Leftrightarrow w \models \{\delta(r)[a]\} \Leftrightarrow w \models \{\delta(r)\}[a] \Leftrightarrow w \models \partial(\{r\})[a]
 \end{aligned}$$

While $\stackrel{\ddagger}{\Rightarrow}$ holds trivially, $\stackrel{\ddagger}{\Leftarrow}$ uses that $\text{take}(1, w) \cdot x \models \delta(r)[a]$ implies that $\text{take}(0, w) \cdot y \models \delta(r)[a]$ holds with $y = \text{take}(1, w) \cdot x$ as a witness.

The final case that we illustrate here is ω -closure r^ω . In this case it is more intuitive to start with the statement $w \models \partial(r^\omega)[a]$. The proof in Lean uses the theorem equivalenceDer as well as the theorem denoteOneStep (and also liftU and liftB that are implicit here). Here the induction hypothesis over RLTL formulas is not applicable. Recall also that $\partial(X\varphi) = \varphi$ and that $v \models X\varphi \Leftrightarrow$

$\text{drop}(1, v) \models \varphi$. The step \star uses the `regexOmegaClosure` theorem in Section 7.6.2.

$$\begin{aligned}
w \models \partial(r^\omega)[a] &\Leftrightarrow w \models \partial(r \diamond \rightarrow \mathbf{X}r^\omega)[a] \\
&\Leftrightarrow w \models ((\text{OneStep}(r) ? r^\omega) \vee \delta(r) \diamond \rightarrow \mathbf{X}r^\omega)[a] \\
&\Leftrightarrow a \models \text{OneStep}(r) \wedge w \models r^\omega \vee w \models \delta(r)[a] \diamond \rightarrow \mathbf{X}r^\omega \\
&\Leftrightarrow a \models \text{OneStep}(r) \wedge w \models r^\omega \vee \exists i : \text{take}(i+1, w) \models \delta(r)[a] \wedge \text{drop}(i, w) \models \mathbf{X}r^\omega \\
&\Leftrightarrow a \models \text{OneStep}(r) \wedge w \models r^\omega \vee \exists i : \text{take}(i+2, a::w) \models r \wedge \text{drop}(i+2, a::w) \models r^\omega \\
&\Leftrightarrow \text{take}(1, a::w) \models r \wedge \text{drop}(1, a::w) \in * r \vee \\
&\quad \exists i : \text{take}(i+2, a::w) \models r \wedge \text{drop}(i+2, a::w) \in * r \\
&\Leftrightarrow \exists i > 0 : \text{take}(i, a::w) \models r \wedge \text{drop}(i, a::w) \in * r \\
&\stackrel{\star}{\Leftrightarrow} a::w \in * r \Leftrightarrow a::w \models r^\omega
\end{aligned}$$

The Lean formalization establishes full confidence in the derivative based semantics of **RLTL**. It also provided feedback that helped to streamline the theory. One particular example is the definition of $\text{OneStep}(r)$ that in its original form implicitly assumed \mathcal{A} to be decidable by depending on satisfiability checking in \mathcal{A} , this assumption turned out to be unnecessary and lead to, both a simpler definition of $\text{OneStep}(r)$ in Section 7.2 as well as a more general theory that applies also to the case when \mathcal{A} is undecidable.

7.7 Omega-Regularity Modulo Theories

Here we lift the classical concept of ω -regular languages [Büchi 1960; McNaughton 1966] as the languages accepted by ABW so as to be modulo \mathcal{A} , and show that $\mathbf{RLTL}^+(\mathcal{A})$ captures ω -regularity modulo \mathcal{A} . We say that $L \subseteq \Sigma^\omega$ is $\omega_{\mathcal{A}}$ -regular if $L = \mathcal{L}(M)$ for some M that is an $NBW_{\mathcal{A}}$. We lift the following result from [Büchi 1960; McNaughton 1966] (see also [Thomas 1990, 1.1. Theorem]) as to be modulo \mathcal{A} . Recall that $\mathcal{L}(R^\omega) = \mathcal{L}(R)^\omega$.

THEOREM 5 ($\omega_{\mathcal{A}}$ -REGULARITY). *A language $L \subseteq \Sigma^\omega$ is $\omega_{\mathcal{A}}$ -regular \Leftrightarrow there exist $n > 0$ and regexes $\{R_i\}_{i=1}^n$ and $\{S_i\}_{i=1}^n$ in $\mathbf{RE}(\mathcal{A})$ such that $L = \bigcup_{i=1}^n \mathcal{L}(R_i) \cdot \mathcal{L}(S_i)^\omega$.*

PROOF. By using Lemma 2 and [McNaughton 1966]. □

Next we show that $\mathbf{RLTL}^+(\mathcal{A})$ captures $\omega_{\mathcal{A}}$ -regularity. We use the following *stepping* lemma of the transition function of any ABW , the proof of which is based on the definition of accepting runs. For all $\phi \in \mathbb{B}^+(Q)$ lift the transition function ϱ to $\hat{\varrho}$ as usual, and for all $a \in \Sigma, u \in \Sigma^*$ let

$$\hat{\varrho}(\epsilon, \phi) \stackrel{\text{DEF}}{=} \phi \quad \hat{\varrho}(a::u, \phi) \stackrel{\text{DEF}}{=} \hat{\varrho}(u, \varrho(\phi)[a]).$$

LEMMA 5. $\forall M \in ABW_{\mathcal{A}}, \phi \in \mathbb{B}^+(Q_M), u \in \Sigma^*, w \in \Sigma^\omega : uw \in \mathcal{L}_M(\phi) \Leftrightarrow w \in \mathcal{L}_M(\hat{\varrho}_M(u, \phi)).$

Recall the construction of the $ABW_{\mathcal{A}} M_\phi$ in Definition 7.6.

THEOREM 6 ($\omega_{\mathcal{A}}$ -REGULARITY OF $\mathbf{RLTL}^+(\mathcal{A})$).

- (1) $L \subseteq \Sigma^\omega$ is $\omega_{\mathcal{A}}$ -regular $\Leftrightarrow \exists \phi \in \mathbf{RLTL}^+(\mathcal{A}) : L = \mathcal{L}(\phi)$
- (2) $\forall \phi \in \mathbf{RLTL}^+(\mathcal{A}) : \mathcal{L}(\phi) = \mathcal{L}(M_\phi)$

PROOF OUTLINE. Statement (1) \Rightarrow is proved by using Theorem 5. Statement (1) \Leftarrow follows from statement (2). Statement (2) is proved by extending the proof in [Vardi 1995] to handle suffix implications. The proof is by induction over ϕ and makes use of Lemma 5. Regex closure formulas are additional base cases that are based on derivative laws of **ERE**.

In order to improve readability we make use of more light-weight notations. In particular, we write $w \models \varphi$ for $w \in \mathcal{L}_M(\varphi)$ and recall that $w_{i..} = \text{take}(i+1, w)$ and $w_{i..} = \text{drop}(i, w)$. The statement (2) is for any $\phi \in \mathbf{RLTL}^+(\mathcal{A})$ equivalently formulated as

$$\forall w \in \Sigma^\omega : w \models \phi \Leftrightarrow w \models \phi$$

We illustrate the key induction case of $\Diamond \rightarrow$ and use of Lemma 5. By construction of $M = M_\phi$ we have that $\partial(q) \cong \varrho_M(q)$ and we let $\varrho = \varrho_M$ below. We also implicitly use the following property:

$$\forall R \in \text{ERE}(\mathcal{A}), a \in \Sigma, u \in \Sigma^* : a \models \text{OneStep}(\hat{\delta}(u, R)) \Leftrightarrow ua \models R$$

Let $w \in \Sigma^\omega$ and $\phi = R \Diamond \rightarrow \psi$. The induction case is to prove that $w \models \phi \Leftrightarrow w \models \phi$ and by the induction hypothesis it holds that $\forall v \in \Sigma^\omega : v \models \psi \Leftrightarrow v \models \psi$. Let $k \in \mathbb{N}$ be some large number.

$$\begin{aligned} w \models R \Diamond \rightarrow \psi &\stackrel{\text{Lma 5}}{\Leftrightarrow} w_{1..} \models \varrho(R \Diamond \rightarrow \psi)[w_0] \\ &\stackrel{(28)}{\Leftrightarrow} w_{1..} \models ((\text{OneStep}(R) ? \partial(\psi)) \vee (\delta(R) \Diamond \rightarrow \psi)) [w_0] \\ &\stackrel{(3)}{\Leftrightarrow} (w_0 \models R \wedge w_{1..} \models \partial(\psi)[w_0]) \vee w_{1..} \models \hat{\delta}(w_0, R) \Diamond \rightarrow \psi \\ &\stackrel{\text{Lma 5}}{\Leftrightarrow} (w_{..0} \models R \wedge w_{0..} \models \psi) \vee w_{1..} \models \hat{\delta}(w_{..0}, R) \Diamond \rightarrow \psi \\ &\stackrel{\dagger}{\Leftrightarrow} \bigvee_{i < k} (w_{..i} \models R \wedge w_{i..} \models \psi) \vee w_{k..} \models \hat{\delta}(w_{..k-1}, R) \Diamond \rightarrow \psi \\ &\stackrel{\ddagger}{\Leftrightarrow} \exists i : w_{..i} \models R \wedge w_{i..} \models \psi \stackrel{\text{IH}}{\Leftrightarrow} \exists i : w_{..i} \models R \wedge w_{i..} \models \psi \stackrel{(23)}{\Leftrightarrow} w \models R \Diamond \rightarrow \psi \end{aligned}$$

In \dagger we repeated the unfolding k times similarly to the previous steps, by using Lemma 5, for some large enough k . In \ddagger we used the key property that all states $r \Diamond \rightarrow \psi$ in M are *nonaccepting* and would cause an infinite branch $(\hat{\delta}(w_{..k+i}, R) \Diamond \rightarrow \psi)_{i \geq 0}$ containing no accepting states if visited infinitely often, i.e., any state $\hat{\delta}(w_{..n}, R) \Diamond \rightarrow \psi$ acts as \perp because it is not visited for any $n \geq k$. \square

Derivatives of R^ω work correctly in Theorem 4, and therefore also when considering $\neg(R^\omega)$ because the semantics is based on RLTL. However, *the complemented derivatives arising from $\neg\partial(R^\omega)$ would in general have incorrect semantics as transitions of an ABW*, as illustrated next.

Example 7.11. Let $\alpha \in \mathcal{A}$, $a \in \llbracket \alpha \rrbracket$, and $\varphi = (\alpha \cdot \alpha)^\omega$, so $a^\omega \models \varphi$. The following derivatives arise from φ by (32) and (28), and where $\text{OneStep}(r)$ shows the predicate used in (28). If we were to start from $\neg\varphi$, by just negating the transition terms, we end up with an erroneous automaton M^{err} :

q	$\partial(q)$	$\text{OneStep}(r)$	$\partial(\neg q)$	M^{err}
φ	$(\alpha ? \alpha \Diamond \rightarrow \mathbf{X}\varphi)$	\perp	$(\alpha ? \neg(\alpha \Diamond \rightarrow \mathbf{X}\varphi) : \top)$	
$\alpha \Diamond \rightarrow \mathbf{X}\varphi$	$(\alpha ? \varphi)$	α	$(\alpha ? \neg\varphi : \top)$	

where the state $\alpha \Box \rightarrow \neg\mathbf{X}\varphi$ is *accepting*, so $a^\omega \in \mathcal{L}(M^{\text{err}})$ while $a^\omega \not\models \neg\varphi$. \boxtimes

8 Related Work

Derivatives. Transition regexes for extended regular expressions [Stanford et al. 2021], a symbolic generalization of [Brzozowski 1964], have been implemented in Z3 [de Moura and Bjørner 2008]. We view $\mathbf{TTerm}(\mathcal{A}, \text{LTL})$ as a symbolic generalization of Vardi's derivatives for LTL [Vardi 1995]. LTL based *monitoring* also uses classical derivatives [Havelund and Roşu 2001; Sen et al. 2003] combined with the decision procedure from [Hsiang 1985]. The relationship between using the tableaux method for LTL [Wolper 1985] vs. linear factors from [Antimirov 1996] were studied in [Sulzmann and Thiemann 2018]. The work in [Antimirov and Mosses 1995] studies Horn-equational reasoning as an alternative to derivatives.

Alternation Elimination. The original alternation elimination algorithm for ABW [Miyano and Hayashi 1984] was studied in-depth by [Boker et al. 2010] with an improved lower bound $O(n2^n)$ for *ordered ABW*.

Non-emptiness of ABW resulting from LTL is studied in [Bloem et al. 1999] through classification into *weak* and *terminal* cases where the problem is easier than in the general case.

The LTL to NBW procedure in [Fritz 2003] starts with an ABW and then computes delayed simulation relations on-the-fly using ϵ -transitions. In the final phase the ABW is translated into an NBW via [Miyano and Hayashi 1984].

The widely used algorithm of [Gerth et al. 1995] uses *tableau* to translate LTL into NBW. It first constructs a *Generalized Büchi automaton* (GBA) that is then, via a well-known encoding, translated into an NBW. Tableau based techniques for LTL were initially studied by Wolper [Wolper 1981, 1983, 1985]. A further extension of the tableau based technique for LTL is introduced in [Couvreur 1999] using on-the-fly expansion of *transition Büchi automata*.

Vardi's [Vardi 1994, Theorem 14, Proof] is the first LTL to ABW construction defined in terms of a step-wise unwinding essentially as derivatives. This construction is not symbolic, as it uses the next element to directly compute a Boolean combination of successor states. The work in [Tsay and Vardi 2021] gives a full construction of LTL to *symbolic co-ABW*. While aspects of the construction are similar to the one in our work, by being based on a symbolic representation of $\rho(q, a)$ – a key difference is in the representation of $\varrho(q)$ as a transition term providing a natural separation of concerns between evaluation of transitions from their target state formulas, that moreover works with *any EBA* \mathcal{A} . The work in [Gastin and Oddoux 2001] modifies Vardi's original construction to produce *very weak co-ABW* instead, prior to the GBA transformation.

The algorithm in [Wulf et al. 2008] defines *symbolic ABW* (sABW) where transition relations are Boolean combinations of literals and successor states, where incremental satisfiability and model checking methods use BDDs [Bryant 1986] both as the alphabet theory and to represent sets of states. For the construction from LTL to sABW the work refers to [Gastin and Oddoux 2001; Vardi 1995].

The first proof that LTL can be translated into Büchi automata appears in [Muller et al. 1988]. It uses weak alternating automata over trees and applies the reduction in [Muller et al. 1986].

The ITE aspect of transition terms is not preserved in the above works and the close relation between states as formulas, even if maintainable in some form in a GBA, gets then lost in the translation to NBW.

Extensions of LTL with Background Theories. In addition to related work discussed in Section 1, *LTL with constraints* [Demri and D'Souza 2007] is a fragment of first-order LTL, where LTL with *Presburger arithmetic* [Demri 2006] is a typical extension of LTL. While many decidable fragments exist, such as CLTL(\mathcal{D}) [Demri and D'Souza 2007], constraint extensions in general lead to undecidability. Further extensions of LTL over infinite domains are introduced in [Grumberg et al. 2012] and subsequently formalized with *generalized register automata* in [Grumberg et al. 2013]. In [Faran and Kupferman 2018], *LTL with arithmetic or LTLA* is studied further with a focus on its decidable and undecidable fragments, where the model-checking problem of the existential fragment is shown to be in PSPACE and the problem is studied also for hierarchical systems.

Recent work in [Geatti et al. 2022] extends LTL with modulo theories over *finite* strings, the theory is generally *undecidable*. The work in [D'Antoni and Veanes 2017b] extends M2L-STR [Henriksen et al. 1995; Klarlund et al. 2002] (WS1S [Büchi 1960] for finite strings) as to be modulo theories. Recently LTL modulo theories has also been studied in [Rodríguez and Sánchez 2023] for *realizability* of temporal specifications where the core algorithms in connection to modulo theories are based on a variant of bitblasting combined with an interpretation back to SMT models. The most recent work in [Heim and Dimitrova 2025] further extends $\text{LTL}\langle\mathcal{A}\rangle$ realizability by allowing *primed* theory variables, which greatly increases scalability for practical applications. Let w be a stream of interpretations over a set $X \cup \{x' \mid x \in X\}$ of theory variables. In terms of $\text{LTL}\langle\mathcal{A}\rangle$ or $\text{RLTL}\langle\mathcal{A}\rangle$ semantics, all *valid* streams $w \in \Sigma^\omega$ must then satisfy the additional semantic condition $\forall i \in \mathbb{N}, x \in X : w_i(x') = w_{i+1}(x)$. For example, $\mathbf{G}(x \leq x')$ implies that x is never decreasing.

In the above works derivatives are not being used and the techniques are in general orthogonal to what we propose here and further extend into $\text{RLTL}\langle\mathcal{A}\rangle$.

Extensions of LTL to ω -Regularity. It has been widely recognized that full ω -regularity is fundamental for general applications of linear temporal properties [Pnueli 1985]. Classical LTL expresses precisely the star-free fragment of ω -regular events [Emerson 1991], e.g., it cannot express ω -regular properties such as $(p \cdot \top)^\omega$ (p holds in all even positions) [Wolper 1983]. Several ω -regular extension of LTL have been proposed: *ETL* [Vardi and Wolper 1994], *LTL with fixpoint operators* [Banieqbal and Barringer 1989], *QPTL* [Sistla et al. 1987] using quantifiers, and *ForSpec* [Armoni et al. 2002] using regular expressions over propositions. *MONA* [Henriksen et al. 1995] also supports LTL and is ω -regular. [Bustan et al. 2005, Proposition 3.14] notes that ω -regularity of ForSpec also carries over to PSL [Eisner and Fisman 2006].

Our extension $\text{RLTL}\langle\mathcal{A}\rangle$ of $\text{LTL}\langle\mathcal{A}\rangle$ with $\Diamond\rightarrow$ and $\{\cdot\}$ was inspired by SPOT [Duret-Lutz 2024; Duret-Lutz et al. 2022] and PSL because both of those operators elegantly admit incremental derivative based unfolding. Originally $\Diamond\rightarrow$ appears in ForSpec as *follows_by* and is also related to the \Diamond -modality in PDL [Fischer and Ladner 1979]. The primary purpose of PSL is for formal specification of concurrent systems. SPOT is a state-of-the-art verification tool that supports a large subset of PSL, including many optimizations that originate from [Cimatti et al. 2008]. RLTL covers a core subset of PSL that is consistent with the semantics of suffix implications in SPOT, but *differs* in one important aspect in regard to *weak closure* (and its negation), namely that, for any nullable regex R , $\{R\} \equiv \top$. For a PSL formula $\{R\}$ the corresponding weak closure in RLTL is $\{R \bowtie \top\}$. This difference is driven by the semantics of derivatives that effectively *enforces* the law $\{R\} \equiv \top$ whenever R is nullable, in order to maintain an algebraically well-behaved system of rules. Thus, some rewrite rules in SPOT, like $\{r*\} \equiv \{r\}$, are *invalid* in RLTL .

RLTL also supports regex *complement* that comes naturally because of the built-in duality law of transition regexes: $\sim(\alpha ? f : g) \equiv (\alpha ? \sim f : \sim g)$ [Stanford et al. 2021, Lemma 4.2]. It is difficult and highly impractical to support \sim by other means, because it would in general require *determinization* of SFAs [Veanes et al. 2010] that is avoided by the duality law that propagates \sim *lazily*.

A fundamental difference is that RLTL lifts (a core subset of) PSL as to be *modulo theories*, while in PSL the atomic regexes α are Boolean combinations of *propositions*.

9 Future Work and Open Problems

The paper presents a self-contained framework with both theory and algorithms to implement symbolic model checking of $\text{RLTL}\langle\mathcal{A}\rangle$. We implemented a prototype in F# where for nonemptiness we used the meticulously presented *nested depth-first cycle detection algorithm* in [Courcoubetis et al. 1992, Algorithm B], that required essentially no modifications. However, it is premature and out of scope to discuss that implementation here. For proper evaluation, one needs a standard that, e.g., combines PSL with SMT-LIB [SMT-LIB 2021], in order to express modulo theories, which, as of today, does not exist. That said, there are many interesting open problems we mention below.

Several kinds of optimizations can be applied. All leaves in DNF can be maintained as *antichains* based on (4) and leverage techniques from [Abdulla et al. 2010; Filiot et al. 2009; Wulf et al. 2008]. At the automata level, we believe that *(bi)simulation* algorithms [Abdulla et al. 2010; Bonchi and Pous 2013; Bustan and Grumberg 2003; Etesami et al. 2005; Fritz 2003; Fritz and Wilke 2002, 2005; Gurumurthy et al. 2002; Mayr and Clemente 2013] can be lifted to $\text{NBW}_{\mathcal{A}}$ similarly to the case of SFAs [D'Antoni and Veanes 2014, 2016, 2017a; Holik et al. 2018] that lift algorithms from [Bustan and Grumberg 2003; Hopcroft 1971; Paige and Tarjan 1987]. Symbolic derivatives can also be relevant for $\text{DBW}_{\mathcal{A}} - \text{NBW}_{\mathcal{A}}$ that omits \vee – and revisit results in [Baarir and Duret-Lutz 2014; Babiak et al. 2012; Baier and Katoen 2008; Diekert et al. 2015; Finkbeiner and Schewe 2005; Kupferman and

Rosenberg 2010; Kurshan 1987; Tourneur and Duret-Lutz 2017]. Implementation of **TTerm** can potentially also use *generic BDDs* [D’Antoni and Veanes 2017b] or *Shannon expansions* [Shannon 1949] for structure sharing. *Dead state detection* in M_R can use [Stanford and Veanes 2023].

To reduce the constant 4 in *product* of $NBW_{\mathcal{A}}$, we believe one can lift the algorithm in [Choueka 1974] ([Kupferman 2018, Theorem 2]) to be modulo \mathcal{A} . For *NBW complementation* [Büchi 1960] ([Kupferman 2018, Theorem 4]), an open question is if derivatives can be used to lift complementation to be modulo \mathcal{A} to support $\neg(R^\omega)$ incrementally.

Counterfactual reasoning in form of causality analysis of ω -regular properties has recently been proposed in [Coenen et al. 2022]. In this context QPTL [Sistla et al. 1987] and HyperQPTL [Beutner and Finkbeiner 2023] are the target formalisms used in the tool developed in [Beutner et al. 2023]. Could derivatives be developed for HyperQPTL and be useful to extend causality analysis as to be modulo background theories?

The *past* operator does not increase expressivity of LTL [Gabbay et al. 1980] but is practically very useful [Armoni et al. 2002; Kupferman and Pnueli 1995; Kupferman et al. 2012; Laroussinie et al. 2002; Lichtenstein et al. 1985]. *Fair correctness* [Varacca and Völzer 2006] also relates to *past*. Derivatives are inherently forward looking, so understanding how to support *past* is intriguing. *Counting* [Laroussinie et al. 2010] with derivatives is another interesting topic. Could derivatives be developed for CTL [Pitman and Pnueli 2018] where *trees* rather than words are the semantic foundation? Derivatives for *finite* trees have been studied in [Attou et al. 2021].

A future challenge is to also formalize the other parts of the theory in *Lean*, including Theorem 1 and Theorem 6, for which the formalization developed here provides a necessary basis. Due to the intricate semantics of $ABW_{\mathcal{A}}$, such a formalization is an invaluable aid, not only for establishing correctness, but also to serve as a platform for formally validating correctness of new rewrite rules and other algorithmic optimizations. Formalizing Theorem 1 in *Lean* is a very difficult task, as it would require formalizing a generalization of [Miyano and Hayashi 1984] that is one of the most technically involved proofs in the classical literature of ω -regularity. A difficult aspect regarding a *Lean* formalization of either Theorem 1 or Theorem 6 is connected to the fact that automata algorithms are cumbersome to express in a proof assistant because, unlike trees or regexes or formulas, graphs are typically not defined inductively.

10 Conclusion

We have shown how transition terms and symbolic derivatives can be used to define a realizable symbolic semantics for (alternating) Büchi automata and linear temporal logic (LTL). The semantics is parameterized by an EBA for the base alphabetic domain, which enables it to apply to ω -languages and infinite alphabets in an algebraically well-defined and precise manner. This framework allows syntactic rewrite rules for LTL extended with regular expressions (RLTL) to be applied *on-the-fly* during alternation elimination, where they simultaneously respect the semantics of RLTL formulas and their alternating Büchi automata. Similarly to the new algorithm \mathcal{A} , there is a rich landscape of further optimizations and algorithms yet to be discovered.

Acknowledgments

We thank Olli Saarikivi for the collaboration that lead to numerous breakthroughs during initial development of the theory. We also thank the anonymous reviewers for their helpful suggestions.

Artifact Availability

The complete *Lean* formalization of Theorem 4, i.e., **theorem** derivation in *Lean*, including all the required background theory, as outlined in Section 7.6, is part of the active github project [Zhuchko 2024]. The artifact [Zhuchko and Ebner 2024] is a snapshot of that project.

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Received 2024-07-08; accepted 2024-11-07